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Research Article

A General Criterion for Unidirectionally Coupled Generalized Chaotic Synchronization with a Desired Manifold

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In this paper, a general criterion for unidirectionally coupled generalized chaotic synchronization between the response system and the original chaotic system in the form of a desired manifold is presented. The expression of the response system constructed with linear error feedback is given, and the validity of the criterion is proved based on the Lyapunov stability theory. Two numerical examples are used to construct unidirectional coupled chaotic systems in the form of different types of manifolds. Numerical simulation is carried out to verify the feasibility of the construction method and the criterion.

1. Introduction

In the field of chaotic synchronization [1], generalized chaotic synchronization [2] is an interesting and important topic. It has been widely applied in many fields such as secure communication [3], mechanical engineering [4], biological system [5], and fluid mixing [6]. According to the two possible different coupling modes—bidirectional and unidirectional—generalized synchronization can be divided into mutual synchronization [7, 8] and master-slave synchronization [9, 10], respectively. In 1995, Rulkov et al. [11] first described the generalized synchronization phenomenon and proposed a method of mutual false nearest neighbors to determine generalized synchronization. Abarbanel et al. [12] proposed the construction of an auxiliary system and used a second identical response system to monitor the generalized synchronization between a drive system and a response system. Subsequently, Kocarev and Parlitz [13] studied the sufficient and necessary conditions for the generalized synchronization of unidirectionally coupled dynamical systems, and they discussed the relationship between the generalized synchronization, predictability, and equivalence of dynamical systems. Boccaletti et al. [14] reviewed the main ideas involved in the synchronization field of chaotic systems and introduced several types of synchronization features in detail.

Over the years, a variety of control methods have been proposed to achieve generalized synchronization between coupled chaotic systems. The linear error feedback control used in this paper is one of the most common control methods. Jiang et al. [15, 16] derived the universal criterion for global chaotic synchronization between two linear unidirectionally coupled chaotic systems. Lü et al. [17] and Yu and Zhang [18] studied the chaotic synchronization between two linear bidirectional coupled chaotic systems, and they proposed some sufficient conditions for global asymptotic synchronization. Wu et al. [19] applied linear error feedback control to the chaotic synchronization of a master-slave generalized Lorentz system, strictly proving the sufficient synchronization criterion of a general linear state error feedback controller. Zhou et al. [20] studied the linear and nonlinear bidirectional coupling synchronization of two hyperchaotic Chen systems. Huang et al. [21] studied the synchronization problem of a time-delay fractional-order chaotic financial system with incommensurate orders, and they obtained sufficient conditions for synchronization by using a linear feedback control strategy and stability theory for fractional-order delayed systems. There are many other

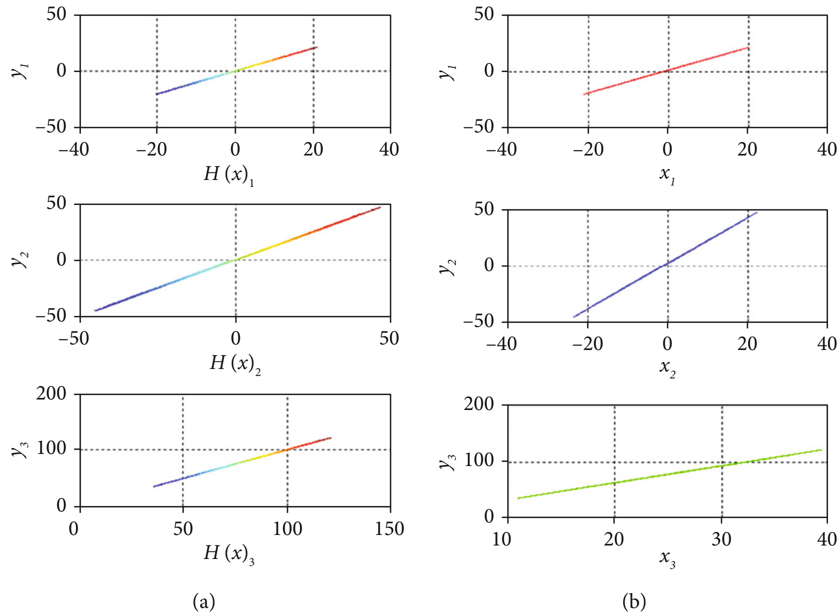


Figure 1: The synchronization relationships of different state variables in the unidirectionally coupled Chen system. (a) $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in complete synchronization. (b) $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in generalized synchronization with respect to the linear manifold $y = Px + Q$.

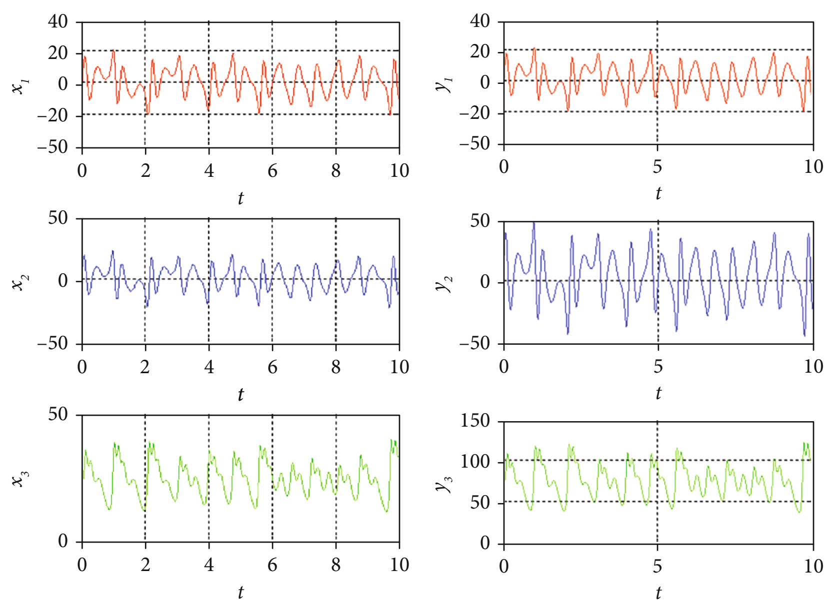


Figure 2: Time histories of different state variables in the unidirectionally coupled Chen system with the linear manifold $y = Px + Q$.

control methods to achieve generalized chaotic synchronization, and some typical ones are introduced in this paper. Ouannas et al. [22] achieved chaotic synchronization between two different chaotic systems by using the nonlinear control law, and they deduced sufficient conditions to ensure the complete synchronization of master-slave models. Lerescu et al. [23] and Guo and Li [24] used OPCL control and adaptive feedback control to synchronize two identical systems from Sprott's simplest chaotic system set. Cheng et al. [25] proposed an adaptive synchronization control law for the Arneodo chaotic system with uncertain parameters and input saturation, and an auxiliary system is used to compensate the synchronization error. Tamba et al. [26] applied the adaptive control to obtain the synchronization of the system with an absolute nonlinearity. Tian et al. [27] proposed an impulse control method for the synchronization of two hyperchaotic Chen circuits with linear or nonlinear delays, and they deduced sufficient conditions for the synchronization of a chaotic system with a time delay. Cui et al. [28] applied the finite time stability theory to design

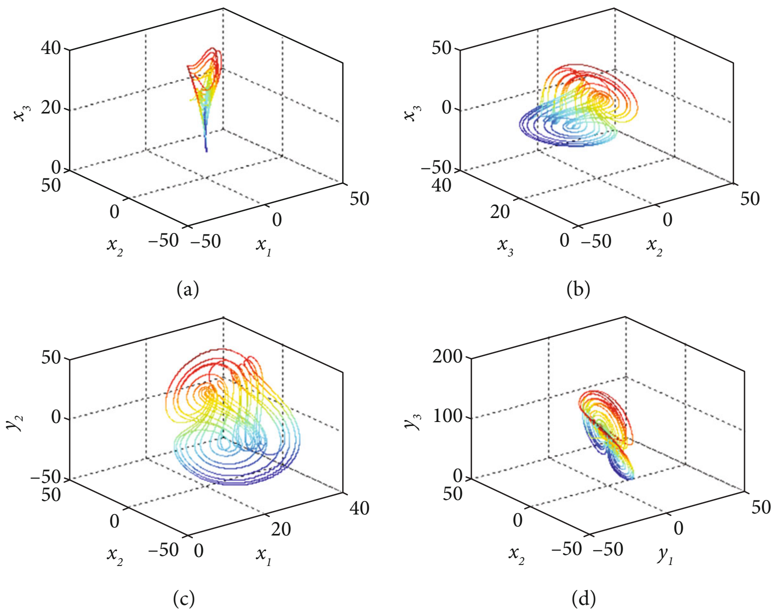


FIGURE 3: The chaotic trajectory of the unidirectionally coupled Chen system with the linear manifold $y = Px + Q$ in 3D phase space: (a) $x_1 - x_2 - x_3$; (b) $x_2 - x_3 - y_1$; (c) $x_3 - y_1 - y_2$; (d) $y_1 - y_2 - y_3$.

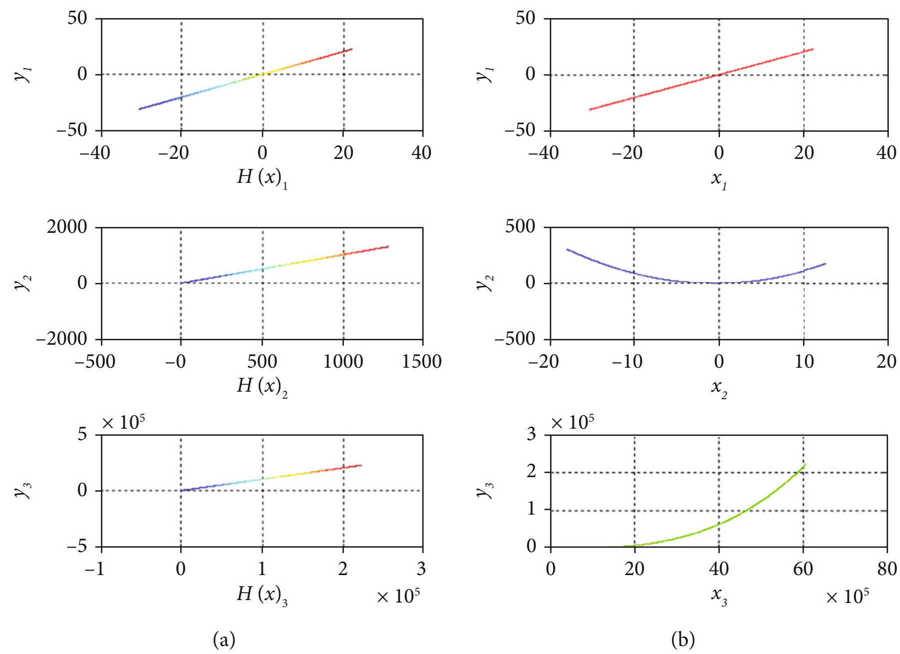


FIGURE 4: The synchronization relationships of different state variables in the unidirectionally coupled Chen system. (a) $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in complete synchronization; (b) $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in generalized synchronization with respect to the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$.

the finite time synchronous controller between the drive system and the corresponding system. Che et al. [29] proposed a robust adaptive neural network (NN) controller to achieve the synchronization of two gap junction coupled chaotic FitzHugh–Nagumo (FHN) neurons under external electrical stimulation. Chaos synchronization is obtained by proper choice of the control parameters. Li and Hernandez [30] proposed a rule-based type-1 fuzzy logic controller (T1-FLC) to synchronize chaotic systems and showed that type-1 fuzzy logic system (T1-FLSs) can effectively deal with uncertainties.

However, in some engineering applications, it is expected that the response system and the drive system can achieve generalized chaotic synchronization with a

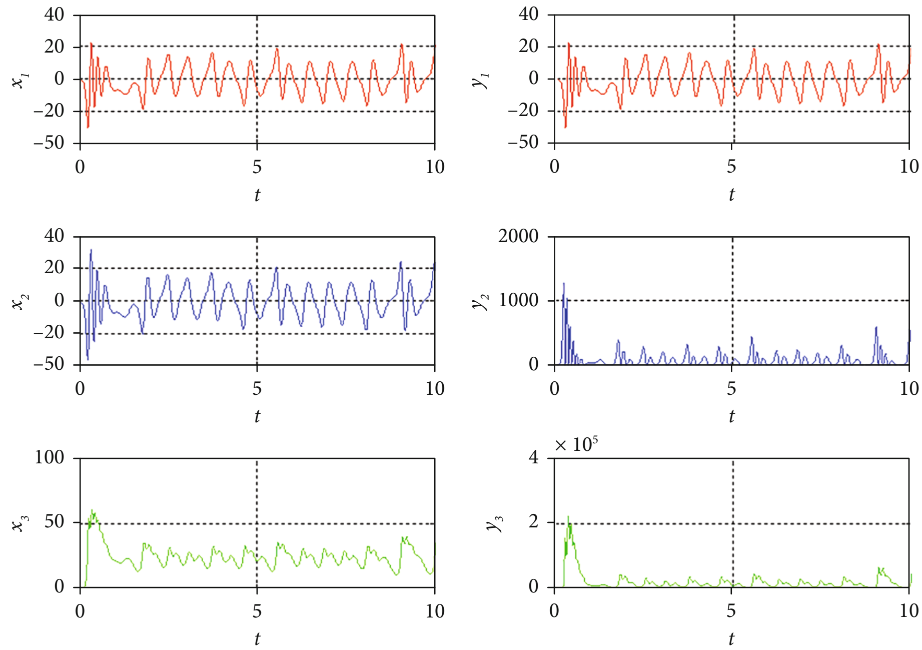


Figure 5: Time histories of different state variables in the unidirectionally coupled Chen system with the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$.

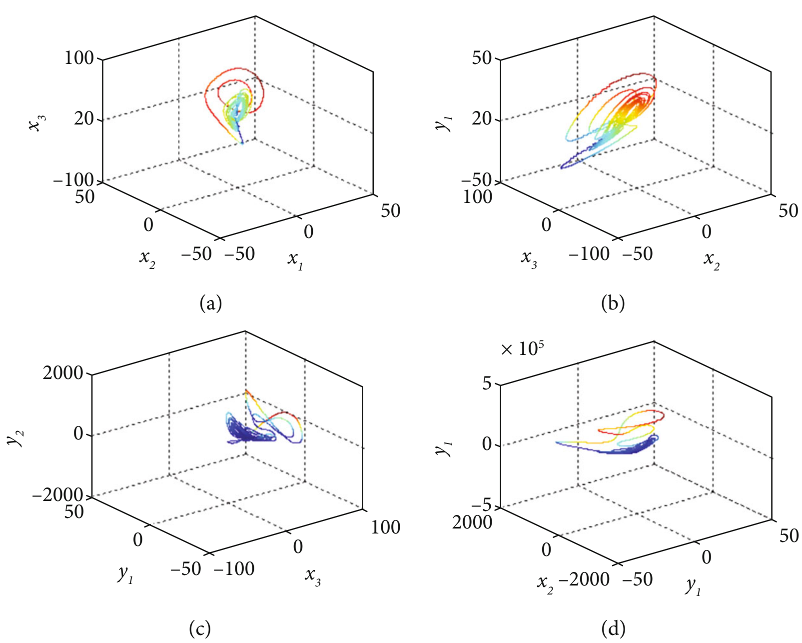
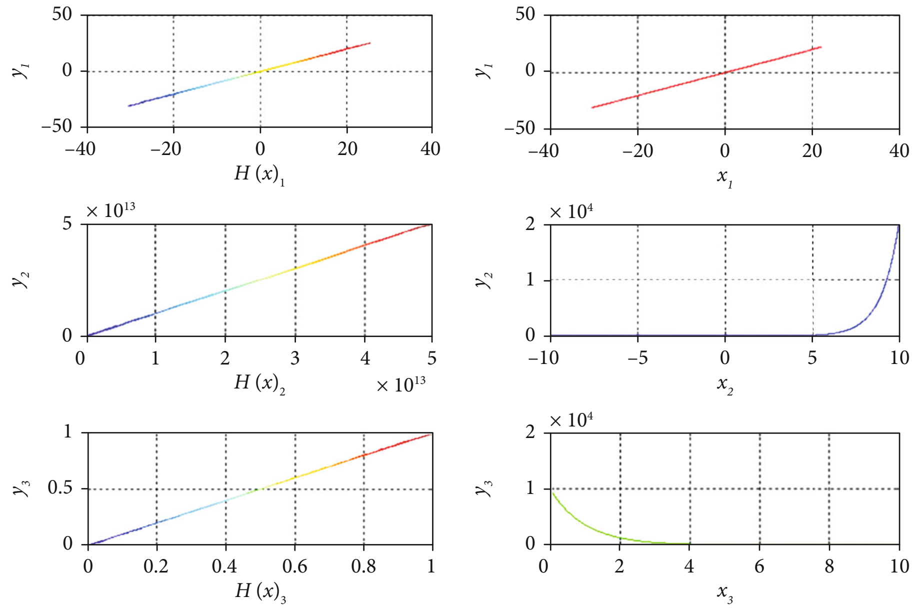


Figure 6: The chaotic trajectory of the unidirectionally coupled Chen system with the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$ in 3D phase space: (a) $x_1 - x_2 - x_3$; (b) $x_2 - x_3 - y_1$; (c) $x_3 - y_1 - y_2$; (d) $y_1 - y_2 - y_3$.

desired manifold. For example, the wide-spectrum characteristics of chaotic signals can be used to conceal the line-spectrum components in acoustic signals of underwater vehicles [31–34]. It is hoped that the modulation of radiation signals can be achieved in a desired manner to conceal the information in the original signals and improve the stealth performance of underwater vehicles. At present, the construction method of a generalized chaotic synchronization system with a desired manifold has yet to be further studied.

Motivated by the above discussions, we propose a general criterion of unidirectionally coupled generalized synchronization between a response system and an original chaotic system with a desired manifold. The expression of



(a) (b)

Figure 7: The synchronization relationships of different state variables in the unidirectionally coupled Chen system. (a) $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in complete synchronization; (b) $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in generalized synchronization with respect to the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$.

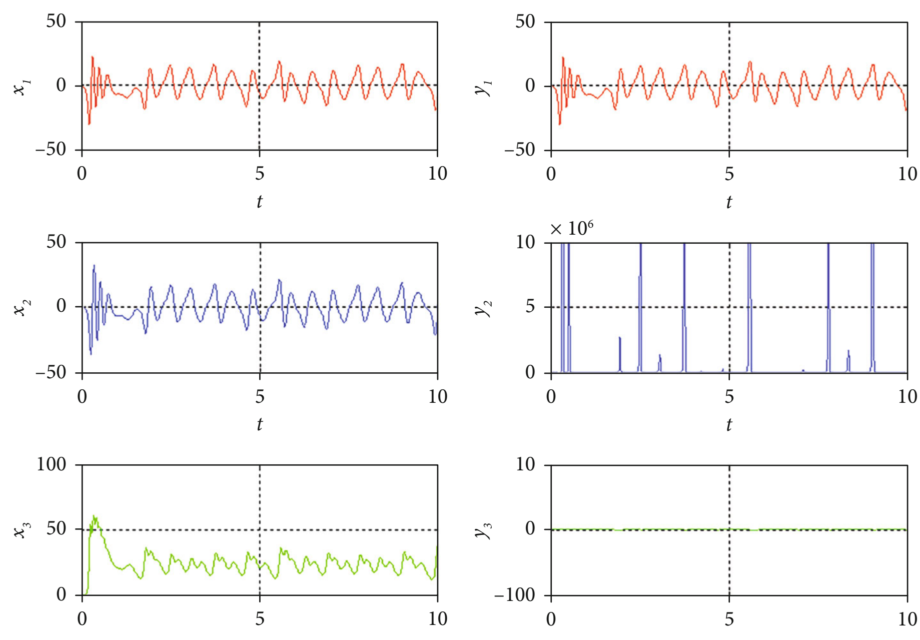


Figure 8: Time histories of different state variables in the unidirectionally coupled Chen system with the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$.

a response system constructed with linear error feedback is provided, and the stability of the zero solution of the error system is proven based on the Lyapunov stability theory. Finally, the validity of the construction method and criterion is verified by numerical simulations.

The structure of this paper is as follows. In Section 2, the definition of generalized synchronization and the expression of the constructed response system are introduced, and the general criterion is proposed and proven theoretically. In Sections 3 and 4, the Chen system and Chua's circuit are taken as examples, unidirectionally coupled chaotic systems are constructed for a linear manifold, a higher order polynomial manifold, and an exponential manifold. The analysis of whether the mapping variables are in complete synchronization with the state variables of the response system is used to judge the generalized synchronization relationship between the original chaotic system and the response system. Then, whether the original chaotic system and the response system achieve generalized synchronization in the form of the desired manifold is determined. Time histories of different state variables of the new system are demonstrated, and the chaotic trajectory of the new system is depicted in three-dimensional (3D) phase space to visualize the dynamical behavior of the system. The conclusions are put forward in the last section.

2. Construction Method and Criterion

Two unidirectionally coupled dynamical systems are considered:

$$\dot{x} = F(x), \tag{1}$$

$$\dot{y} = G(x, y), \tag{2}$$

where $x = (x_1, \cdots, x_n)^T \in R^n$, $y = (y_1, \cdots, y_m)^T \in R^m$, the differentiable vector function $F(x) = (f_1(x), \cdots, f_n(x))^T \in R^n$, and $G(x, y) = (g_1(x, y), \cdots, g_n(x, y))^T \in R^m$.

Definition 1. If there is a known differential homeomorphic map $H : R^n \longrightarrow R^m$, and there are subsets $B = B_x \times B_y \subset R^n \times R^m$ such that all trajectories $(x(0), y(0)) \in B$ in dynamical systems (1) and (2) satisfy $\lim_{t\longrightarrow\infty} \|y(t, y(0)) - H(x(t, x(0)))\| = 0$, systems (1) and (2) are determined to have achieved generalized synchronization with the manifold $y = H(x)$.

For chaotic system (1), the response system is constructed through linear error feedback. Then, system (2) can be rewritten as

$$\dot{y} = DH(x) \cdot F(x) + K(y - H(x)), \tag{3}$$

where $DH(x)$ is the Jacobian matrix of $H(x)$, and K is the diagonal matrix containing the coupling coefficient, denoted by $K = \text{diag}\,(k_1, k_2, \cdots, k_m)$, $k_i \in R$, and $i = 1, 2, \cdots, m$.

Theorem 2. *If the coupling matrix K is chosen such that*

$$\lambda_i < 0, \quad i = 1, 2, \cdots, m, \tag{4}$$

where λ_i is the eigenvalue of the matrix $K^T P + PK$, and P is a positive definite symmetric constant matrix, then unidirectionally coupled dynamical systems (1) and (3) achieve generalized synchronization with the manifold $y = H(x)$.

Proof of Theorem 1. Let the error term $e = y - H(x)$, the equation of the error system is obtained from equations (1) and (3) as follows:

$$\dot{e} = \dot{y} - DH(x)\dot{x} = DH(x) \cdot F(x) + K(y - H(x)) - DH(x) \cdot F(x) = Ke. \tag{5}$$

The Lyapunov function is chosen, given by

$$V = e^T Pe, \tag{6}$$

where P is a positive definite symmetric constant matrix. Then,

$$\dot{V} = \dot{e}^T Pe + e^T P\dot{e} = e^T K^T Pe + e^T PKe = e^T Q_1 e, \tag{7}$$

where $Q_1 = K^T P + PK$. □

Since $Q_1 = {Q_1}^T$, and Q_1 is a real symmetric matrix, let $Q_1 = {U_1}^T \Lambda U_1$, where U_1 is an orthogonal matrix, and $\Lambda = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_m)$. Then,

$$\dot{V} = e^T Q_1 e = e^T {U_1}^T \Lambda U_1 e = {e_1}^T \Lambda e_1 < 0, \tag{8}$$

where $e_1 = U_1 e$. According to the Lyapunov stability theory, $\lim_{t\longrightarrow\infty} \|e(t)\| = 0$, the zero solution of the error system (5) is asymptotically stable; so, systems (1) and (3) achieve generalized synchronization with the manifold $y = H(x)$.□.

Remark 3. If $P = I$, then K is a negative definite matrix, which meets the design requirements.

3. Chen System

The chaotic Chen system [35] can be expressed by the following nonlinear ordinary differential equation:

$$\begin{cases} \dfrac{dx_1}{dt} = a(x_2 - x_1), \\ \dfrac{dx_2}{dt} = (c - a)x_1 - x_1 x_3 + c x_2, \\ \dfrac{dx_3}{dt} = x_1 x_2 - b x_3, \end{cases} \tag{9}$$

where $a = 35$, $b = 3$, and $c = 28$.

3.1. Linear Manifold. The mapping $H(x) = Px + Q$ is chosen; $P = [1\,0\,0\,;0\,2\,0\,;0\,0\,3]$, and $Q = [1\,2\,3]^T$. At this time, $DH(x) = P$. For system (9), the initial condition is set as $x(0) = (-10, 0, 27)^T$ and $y(0) = H(x(0)) = (-9, 2, 84)^T$, and $K = \text{diag}\,(k_1, k_2, k_3) = \text{diag}\,(-0.5, -0.5, -0.5)$ is selected. According to equation (3), linear error feedback is used to construct

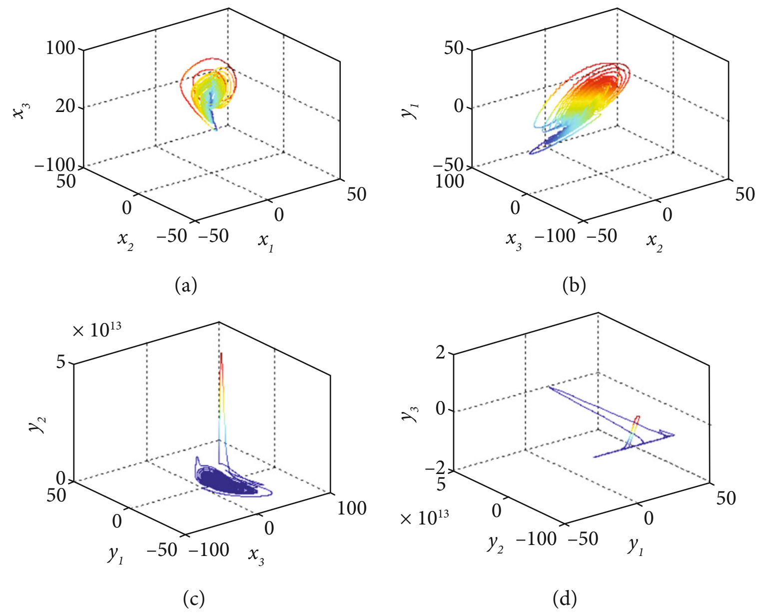


Figure 9: The chaotic trajectory of the unidirectionally coupled Chen system with the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$ in 3D phase space: (a) $x_1 - x_2 - x_3$; (b) $x_2 - x_3 - y_1$; (c) $x_3 - y_1 - y_2$; (d) $y_1 - y_2 - y_3$.

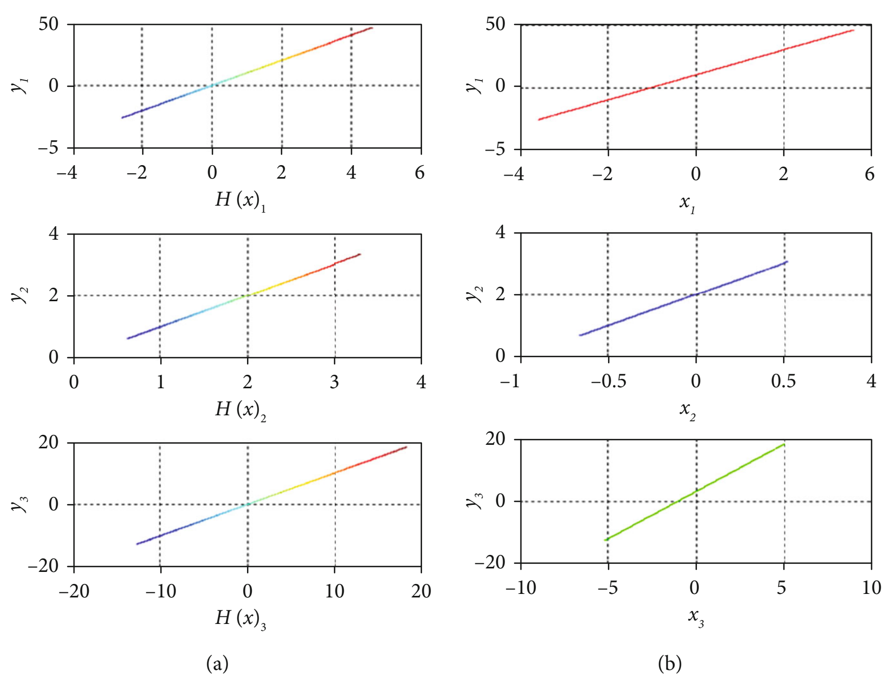


Figure 10: The synchronization relationships of different state variables in the unidirectionally coupled Chua's circuit. (a) $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in complete synchronization; (b) $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in generalized synchronization with respect to the linear manifold $y = Px + Q$.

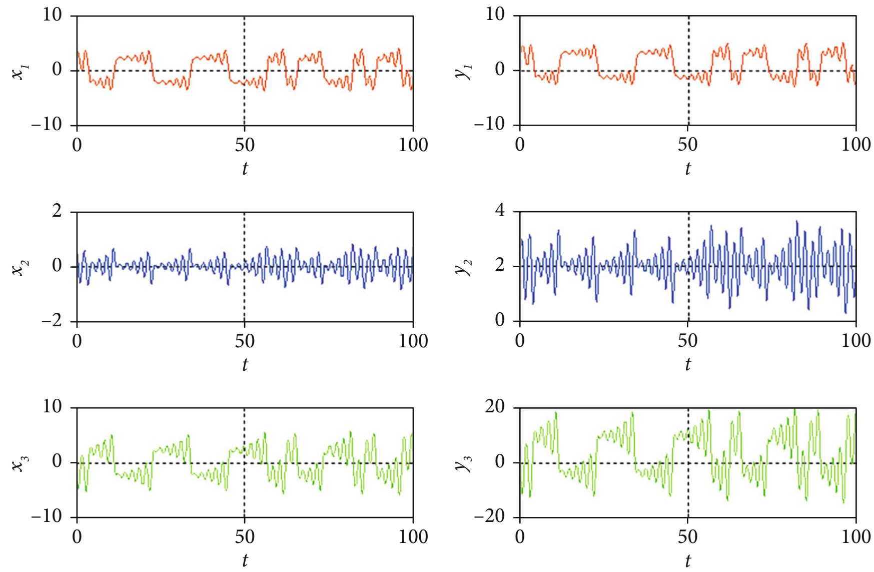


FIGURE 11: Time histories of different state variables in the unidirectionally coupled Chua's circuit with the linear manifold $y = Px + Q$.

the response system:

$$\begin{aligned}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix} &= \begin{bmatrix}1 & & \\ & 2 & \\ & & 3\end{bmatrix}\begin{bmatrix}a(x_2-x_1)\\ (c-a)x_1 - x_1x_3 + cx_2\\ x_1x_2 - bx_3\end{bmatrix} + \begin{bmatrix}k_1 & & \\ & k_2 & \\ & & k_3\end{bmatrix}\begin{bmatrix}y_1 - H(x)_1\\ y_2 - H(x)_2\\ y_3 - H(x)_3\end{bmatrix}\\ &= \begin{bmatrix}a(x_2-x_1) + k_1[y_1 - (x_1+1)]\\ 2[(c-a)x_1 - x_1x_3 + cx_2] + k_2[y_2 - (2x_2+2)]\\ 3(x_1x_2 - bx_3) + k_3[y_3 - (3x_3+3)]\end{bmatrix}.\end{aligned} \tag{10}$$

It can be seen from Figure 1 that the mapping variable $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ of the Chen system and the state variable $\boldsymbol{y} = (y_1, y_2, y_3)^T$ of the response system are in complete synchronization, and the state variable $x = (x_1, x_2, x_3)^T$ of the Chen system and the state variable $y = (y_1, y_2, y_3)^T$ of the response system achieve generalized synchronization with the linear manifold $y = Px + Q$. Figure 2 shows time histories of different state variables in the unidirectionally coupled Chen system with the linear manifold $y = Px + Q$. The chaotic trajectory of the constructed system in 3D phase space is shown in Figure 3. In summary, a unidirectionally coupled generalized chaotic synchronization system based on the Chen system is constructed for the linear manifold $y = Px + Q$.

3.2. Higher-Order Polynomial Manifold. The mapping $H(x) = [x_1, x_2 + x_2^2, x_3^3]^T$ is chosen, and at this time, $DH(x) = \mathrm{diag}\,(1, 1 + 2x_2, 3x_3^2)$. For system (9), the initial condition is set as $x(0) = (1, 0, 0)^T$ and $y(0) = H(x(0)) = (1, 0, 0)^T$, and $K = \mathrm{diag}\,(k_1, k_2, k_3) = \mathrm{diag}\,(-1,-1,-1)$ is selected.

Linear error feedback is used to construct the response system:

$$\begin{aligned}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix} &= \begin{bmatrix}1 & & \\ & 1+2x_2 & \\ & & 3x_3^2\end{bmatrix}\begin{bmatrix}a(x_2-x_1)\\ (c-a)x_1 - x_1x_3 + cx_2\\ x_1x_2 - bx_3\end{bmatrix} + \begin{bmatrix}k_1 & & \\ & k_2 & \\ & & k_3\end{bmatrix}\begin{bmatrix}y_1 - H(x)_1\\ y_2 - H(x)_2\\ y_3 - H(x)_3\end{bmatrix}\\ &= \begin{bmatrix}a(x_2-x_1) + k_1(y_1 - x_1)\\ (1+2x_2)[(c-a)x_1 - x_1x_3 + cx_2] + k_2\left[y_2 - \left(x_2 + x_2^2\right)\right]\\ 3x_3^2(x_1x_2 - bx_3) + k_3\left(y_3 - x_3^3\right)\end{bmatrix}.\end{aligned} \tag{11}$$

It can be seen from Figure 4 that the mapped variables $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ of the Chen system and the state variables $y = (y_1, y_2, y_3)^T$ of the response system are in complete synchronization, and the state variable $x = (x_1, x_2, x_3)^T$ of the Chen system and the state variable $y = (y_1, y_2, y_3)^T$ of the response system achieve generalized synchronization with the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$. Figure 5 shows time histories of different state variables in the unidirectionally coupled Chen system with the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$. The chaotic trajectory of the constructed

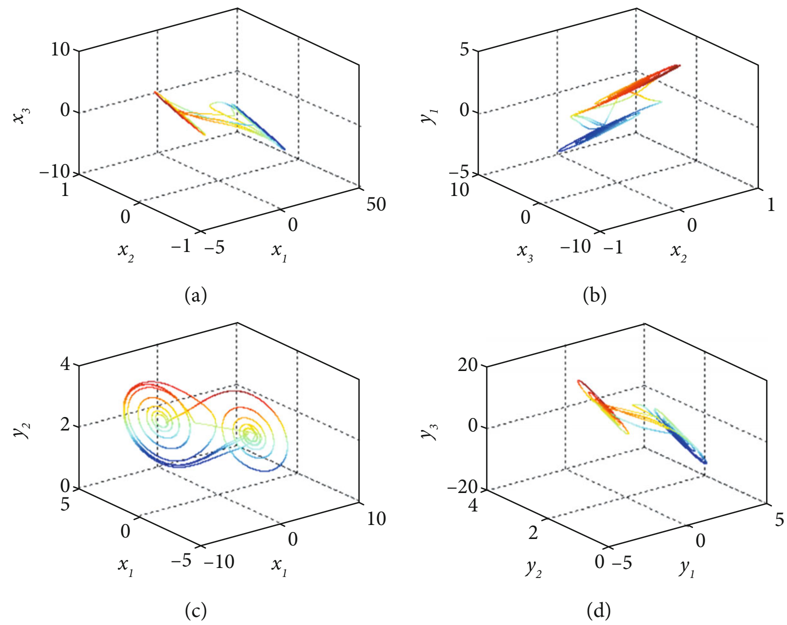


FIGURE 12: The chaotic trajectory of the unidirectionally coupled Chua's circuit with the linear manifold $y = Px + Q$ in 3D phase space: (a) $x_1 - x_2 - x_3$; (b) $x_2 - x_3 - y_1$; (c) $x_3 - y_1 - y_2$; (d) $y_1 - y_2 - y_3$.

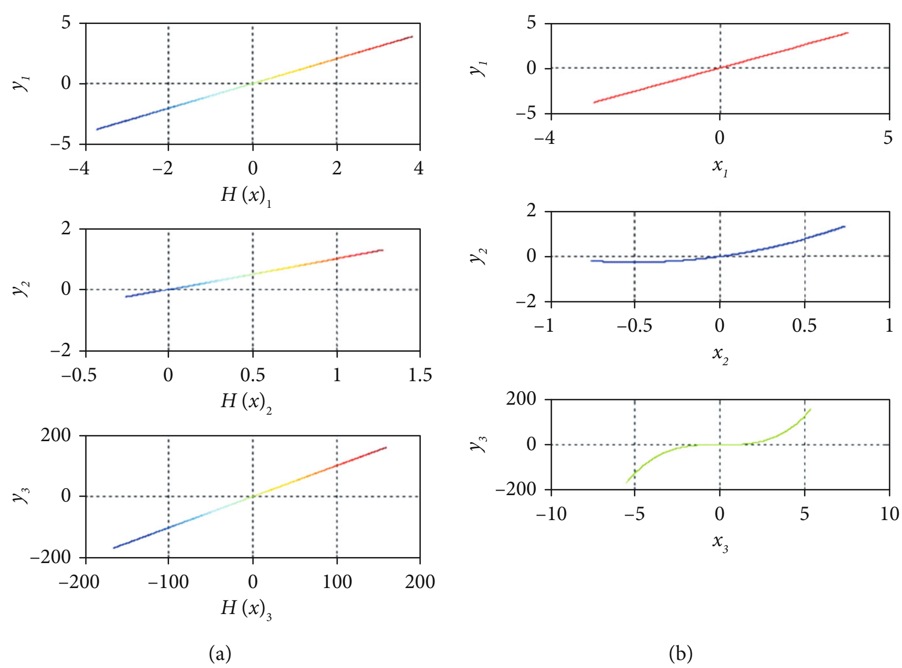


FIGURE 13: The synchronization relationships of different state variables in the unidirectionally coupled Chua's circuit. (a) $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in complete synchronization; (b) $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in generalized synchronization with respect to the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$.

system in the 3D phase space is shown in Figure 6. In summary, a unidirectionally coupled generalized chaotic synchronization system based on the Chen system is constructed for the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$.

3.3. Exponential Manifold. The mapping $H(x) = [x_1, e^{x_2}, e^{-x_3}]^T$ is chosen, and at this time, $DH(x) = \mathrm{diag}\,(x_1, e^{x_2}, -e^{-x_3})$. For system (9), the initial condition is set as $x(0) = (1, 0, 0)^T$ and $y(0) = H(x(0)) = (1, 1, 1)^T$, and $K = \mathrm{diag}\,(k_1, k_2, k_3) = \mathrm{diag}\,(-1, -1, -1)$ is selected.

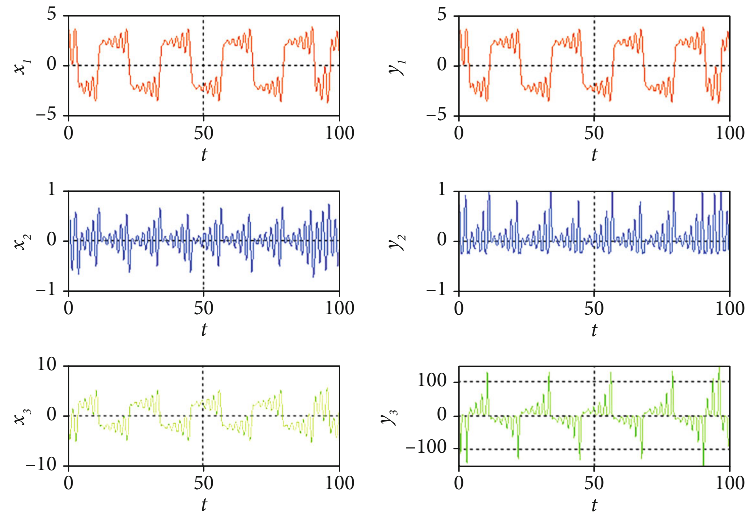


FIGURE 14: Time histories of different state variables in the unidirectionally coupled Chua's circuit with the high-order polynomial manifold $y=[x_1, x_2+x_2^2, x_3^3]^T$.

Linear error feedback is used to construct the response system:

$$\begin{aligned}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix} &= \begin{bmatrix}x_1 & & \\ & e^{x_2} & \\ & & -e^{-x_3}\end{bmatrix}\begin{bmatrix}a(x_2-x_1)\\ (c-a)x_1-x_1x_3+cx_2\\ x_1x_2-bx_3\end{bmatrix} + \begin{bmatrix}k_1 & & \\ & k_2 & \\ & & k_3\end{bmatrix}\begin{bmatrix}y_1-H(x)_1\\ y_2-H(x)_2\\ y_3-H(x)_3\end{bmatrix}\\ &= \begin{bmatrix}ax_1(x_2-x_1)+k_1(y_1-x_1)\\ e^{x_2}[(c-a)x_1-x_1x_3+cx_2]+k_2(y_2-e^{x_2})\\ -e^{-x_3}[x_1x_2-bx_3]+k_3(y_3-e^{-x_3})\end{bmatrix}.\end{aligned} \tag{12}$$

It can be seen from Figure 7 that the mapped variables $H(x)=(H(x)_1, H(x)_2, H(x)_3)^T$ of the Chen system and the state variables $y=(y_1, y_2, y_3)^T$ of the response system are in complete synchronization, and the state variable $x=(x_1, x_2, x_3)^T$ of the Chen system and the state variable $y=(y_1, y_2, y_3)^T$ of the response system achieve generalized synchronization with the exponential manifold $y=[x_1, e^{x_2}, e^{-x_3}]^T$. Figure 8 shows time histories of different state variables in the unidirectionally coupled Chen system with the exponential manifold $y=[x_1, e^{x_2}, e^{-x_3}]^T$. The chaotic trajectory of the constructed system in the 3D phase space is shown in Figure 9. In summary, a unidirectionally coupled generalized chaotic synchronization system based on the Chen system is constructed for the exponential manifold $y=[x_1, e^{x_2}, e^{-x_3}]^T$. In addition, it should be noted that the amplitude of the state variable x_3 of the Chen system is always much greater than zero; so, the amplitude of the state variable y_3 of the response system attenuate sharply to zero and remains unchanged. This feature has positive significance for concealing acoustic signals and improving the acoustic stealth performance of underwater vehicles.

4. Chua's Circuit

The chaotic Chua's circuit [36] can be expressed by the following nonlinear ordinary differential equation:

$$\begin{cases}\dfrac{dx_1}{dt}=\alpha(x_2-x_1)-\alpha f(x_1),\\ \dfrac{dx_2}{dt}=x_1-x_2+x_3\\ \dfrac{dx_3}{dt}=-\beta x_2,\end{cases}, \quad f(x_1)=bx_1+\frac{1}{2}(a-b)(|x_1+1|-|x_1-1|), \tag{13}$$

where $\alpha=9.78$, $\beta=14.97$, $a=-1.31$, and $b=-0.75$.

4.1. Linear Manifold. The mapping $H(x)=Px+Q$ is chosen; $P=[1\,0\,0\,;0\,2\,0\,;0\,0\,3]$, and $Q=[1\,2\,3]^T$, and at this time, $D\,H(x)=P$. For system (13), the initial condition is set as $x(0)=(1,0,0)^T$ and $y(0)=H(x(0))=(2,2,3)^T$, and $K=\operatorname{diag}(k_1,k_2,k_3)=\operatorname{diag}(-0.5,-0.5,-0.5)$ is selected. The response system is constructed according to equation (3):

$$\begin{aligned}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix} &= \begin{bmatrix}1 & & \\ & 2 & \\ & & 3\end{bmatrix}\begin{bmatrix}\alpha(x_2-x_1)-\alpha f(x_1)\\ x_1-x_2+x_3\\ -\beta x_2\end{bmatrix} + \begin{bmatrix}k_1 & & \\ & k_2 & \\ & & k_3\end{bmatrix}\begin{bmatrix}y_1-H(x)_1\\ y_2-H(x)_2\\ y_3-H(x)_3\end{bmatrix}\\ &= \begin{bmatrix}\alpha(x_2-x_1)-\alpha f(x_1)+k_1[y_1-(x_1+1)]\\ 2(x_1-x_2+x_3)+k_2[y_2-(2x_2+2)]\\ -3\beta x_2+k_3[y_3-(3x_3+3)]\end{bmatrix}.\end{aligned} \tag{14}$$

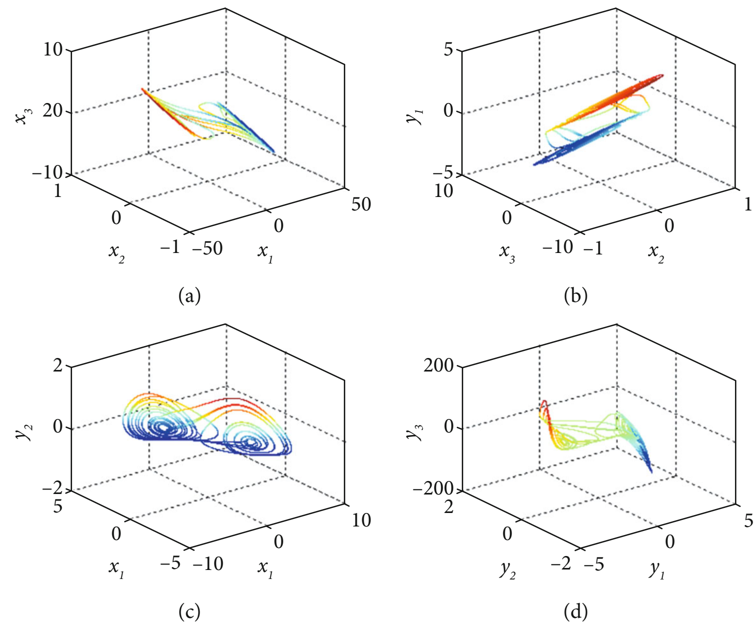


Figure 15: The chaotic trajectory of the unidirectionally coupled Chua's circuit with the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$ in 3D phase space: (a) $x_1 - x_2 - x_3$; (b) $x_2 - x_3 - y_1$; (c) $x_3 - y_1 - y_2$; (d) $y_1 - y_2 - y_3$.

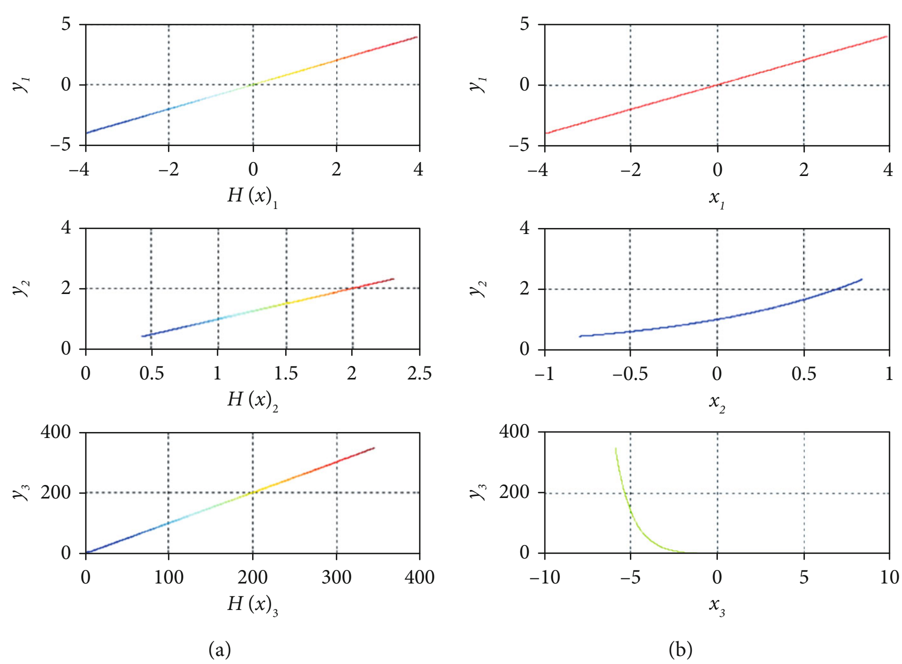


Figure 16: The synchronization relationships of different state variables in the unidirectionally coupled Chua's circuit. (a) $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in complete synchronization; (b) $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ are in generalized synchronization with respect to the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$.

It can be seen from Figure 10 that the mapping variable $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ of Chua's circuit and the state variable $y = (y_1, y_2, y_3)^T$ of the response system are in complete synchronization, and the state variable $x = (x_1, x_2, x_3)^T$ of Chua's circuit and the state variable $y = (y_1, y_2, y_3)^T$ of the response system achieve generalized synchronization with the linear manifold $y = Px + Q$. Figure 11 shows time histories of different state variables in the unidirectionally coupled Chua's circuit with the linear manifold $y = Px + Q$. The chaotic trajectory of the constructed system in 3D phase space is shown in Figure 12.

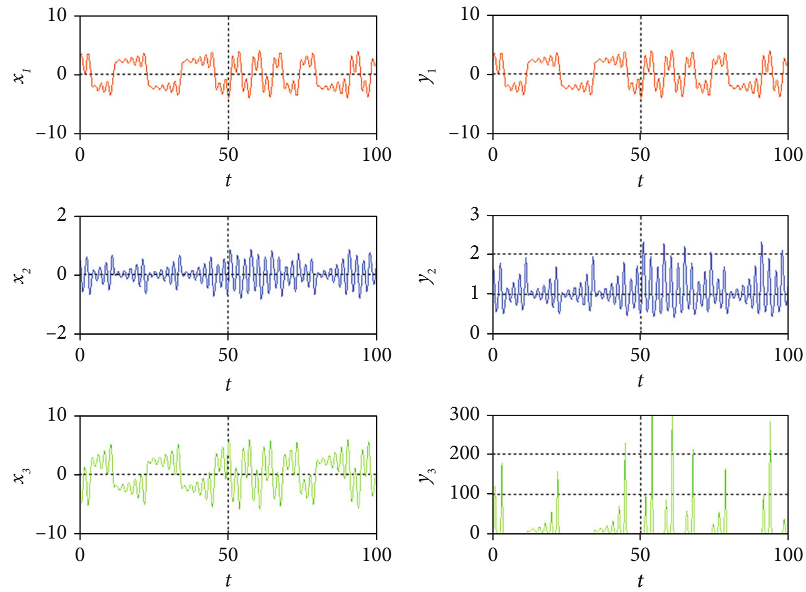


FIGURE 17: Time histories of different state variables in the unidirectionally coupled Chua's circuit with the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$.

In summary, a unidirectionally coupled generalized chaotic synchronization system based on Chua's circuit is constructed for the linear manifold $y = Px + Q$.

4.2. Higher-Order Polynomial Manifold. The mapping $H(x) = [x_1, x_2 + x_2^2, x_3^3]^T$ is chosen, and at this time, $DH(x) = \text{diag}\,(1, 1 + 2x_2, 3x_3^2)$. For system (13), the initial condition is set as $x(0) = (1, 0, 0)^T$ and $y(0) = H(x(0)) = (1, 0, 0)^T$, and $K = \text{diag}\,(k_1, k_2, k_3) = \text{diag}\,(-1,-1,-1)$ is selected.

Linear error feedback is used to construct the response system:

$$\begin{aligned}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix} &= \begin{bmatrix}1 & & \\ & 1+2x_2 & \\ & & 3x_3^2\end{bmatrix}\begin{bmatrix}\alpha(x_2-x_1)-\alpha f(x_1)\\ x_1-x_2+x_3\\ -\beta x_2\end{bmatrix} \\ &\quad + \begin{bmatrix}k_1 & & \\ & k_2 & \\ & & k_3\end{bmatrix}\begin{bmatrix}y_1-H(x)_1\\ y_2-H(x)_2\\ y_3-H(x)_3\end{bmatrix} \\ &= \begin{bmatrix}\alpha(x_2-x_1)-\alpha f(x_1)+k_1(y_1-x_1)\\ (1+2x_2)(x_1-x_2+x_3)+k_2\left[y_2-\left(x_2+x_2^2\right)\right]\\ -3\beta x_3^2x_2+k_3\left(y_3-x_3^3\right)\end{bmatrix}.\end{aligned} \tag{15}$$

It can be seen from Figure 13 that the mapped variables $H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ of Chua's circuit and the state variables $y = (y_1, y_2, y_3)^T$ of the response system are in complete synchronization, and the state variable $x = (x_1, x_2, x_3)^T$ of Chua's circuit and the state variable $y = (y_1, y_2, y_3)^T$ of the response system achieve generalized synchronization with the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$. Figure 14 shows time histories of different state variables in the unidirectionally coupled Chua's circuit with the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$. The chaotic trajectory of the constructed system in the 3D phase space is shown in Figure 15. In summary, a unidirectionally coupled generalized chaotic synchronization system based on Chua's circuit is constructed for the high-order polynomial manifold $y = [x_1, x_2 + x_2^2, x_3^3]^T$.

4.3. Exponential Manifold. The mapping $H(x) = [x_1, e^{x_2}, e^{-x_3}]^T$ is chosen, and at this time, $DH(x) = \text{diag}\,(x_1, e^{x_2}, -e^{-x_3})$. For system (13), the initial condition is set as $x(0) = (1, 0, 0)^T$ and $y(0) = H(x(0)) = (1, 1, 1)^T$, and $K = \text{diag}\,(k_1, k_2, k_3) = \text{diag}\,(-1,-1,-1)$ is selected.

Linear error feedback is used to construct the response system:

$$\begin{aligned}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix} &= \begin{bmatrix}x_1 & & \\ & e^{x_2} & \\ & & -e^{-x_3}\end{bmatrix}\begin{bmatrix}\alpha(x_2-x_1)-\alpha f(x_1)\\ x_1-x_2+x_3\\ -\beta x_2\end{bmatrix} \\ &\quad + \begin{bmatrix}k_1 & & \\ & k_2 & \\ & & k_3\end{bmatrix}\begin{bmatrix}y_1-H(x)_1\\ y_2-H(x)_2\\ y_3-H(x)_3\end{bmatrix} \\ &= \begin{bmatrix}\alpha(x_2-x_1)-\alpha f(x_1)+k_1(y_1-x_1)\\ e^{x_2}(x_1-x_2+x_3)+k_2(y_2-e^{x_2})\\ e^{-x_3}\beta x_2+k_3(y_3-e^{-x_3})\end{bmatrix}.\end{aligned} \tag{16}$$

It can be seen from Figure 16 that the mapped variables

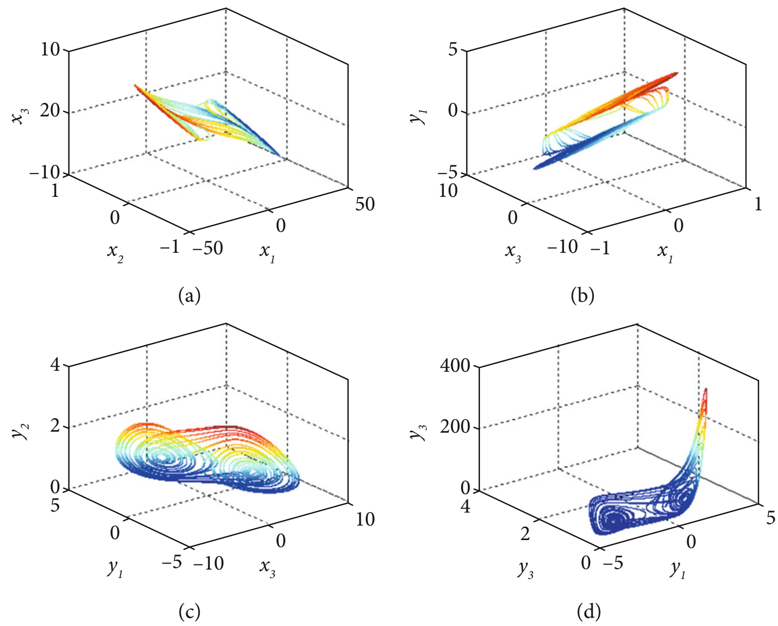


FIGURE 18: The chaotic trajectory of the unidirectionally coupled Chua's circuit with the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$ in 3D phase space: (a) $x_1 - x_2 - x_3$; (b) $x_2 - x_3 - y_1$; (c) $x_3 - y_1 - y_2$; (d) $y_1 - y_2 - y_3$.

$H(x) = (H(x)_1, H(x)_2, H(x)_3)^T$ of Chua's circuit and the state variables $y = (y_1, y_2, y_3)^T$ of the response system are in complete synchronization, and the state variable $x = (x_1, x_2, x_3)^T$ of Chua's circuit and the state variable $y = (y_1, y_2, y_3)^T$ of the response system achieve generalized synchronization with the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$. Figure 17 shows time histories of different state variables in the unidirectionally coupled Chua's circuit with the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$. The chaotic trajectory of the constructed system in the 3D phase space is shown in Figure 18. In summary, a unidirectionally coupled generalized chaotic synchronization system based on the Chua's circuit is constructed for the exponential manifold $y = [x_1, e^{x_2}, e^{-x_3}]^T$.

5. Conclusions

In order to modulate the signal in an ideal way, this paper proposes a criterion for constructing a unidirectional coupled system based on the original system, inspired by the linear error feedback method, so that the response system and the drive system achieve generalized chaotic synchronization in the form of a desired manifold. The criterion gives an expression for a unidirectional coupled system constructed by linear error feedback. The validity and feasibility of the criterion are demonstrated by theoretical analysis and numerical simulation. The simulation shows that the mapped variables of the original chaotic system and the state variables of the response system are completely synchronized, and the state variables of the original chaotic system and the state variables of the response system achieve generalized synchronization in the form of the desired manifold. The dynamic behavior of the new system is visualized by describing the time histories of different state variables and the chaotic trajectories of the system. This paper extends the research results of generalized synchronization between coupled chaotic systems, which is an important reference for the application of generalized chaotic synchronization in practical systems.

Recently, fractional-order systems and fractional-order control have received significant attention in academia and industry. They provide increased flexibility over integer-order systems, allow more accurate modeling of complex systems and meet more challenging control requirements. Therefore, the method and criterion for chaotic synchronization of fractional-order nonlinear systems in the form of the desired manifold may be one of the future research projects.

Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there is no conflicts of interest regarding the publication of this paper.

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