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On a Local Equality of Distributions and Relation of Jumps of Distributions with Fourier Series

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Abstract

Distribution theory has an important role in applied mathematics. Firstly, in the introduction part of this paper we will give some general notations, definitions and results in distribution theory, as analytic representation of distribution, distributional jump behavior, distributional symmetric jump behavior, tempered distributions, formulas for the jump of distributions in terms of Fourier series and global equality in distributional sense. Then in final part we will state two results, the first one has to do on local equality of two distributions and the second one states if a $2\pi-$periodic distribution has jump behavior at point, then exists the relation of jumps of distributions with Fourier series in a subset of the upper half-plane.

Keywords: Distributions, distributional jump behavior, point value, support, Schwartz space.

1 Introduction

Firstly, let us give some introductory concepts. With P^n we denote the subset of R^n, which elements have nonnegative integers coordinates. For a function

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$f, f:\Omega \to \mathrm{C}^n, \Omega \subseteq \mathrm{R}^n,\ k=(k_1,k_2,\ ,k_n),\ k_j \in \mathrm{N}\cup\{0\}$, $x \in \Omega$, with $f^{(k)}(x)$ we denote the differential operator

$$f^{(k)}(x)=\frac{\partial^{|k|}}{\partial_{x_1}^{k_1}...\partial_{x_n}^{k_n}}f(x)=\partial^k f(x), \qquad |k|=k_1+k_2+...+k_n.$$

With $C^\infty(\mathrm{R}^n)$ is denoted the space of all complex valued infinitely differentiable functions on R^n and $C_0^\infty(\mathrm{R}^n)$ denotes the subspace of $C^\infty(\mathrm{R}^n)$ that consists of those functions of $C^\infty(\mathrm{R}^n)$ which have compact support.

Definition 1. The support of f is the closure of the set $x \in \Omega$, of points for which f is different from zero $(f(x) \neq 0)$, and is denoted by supp f .

With D we denote the space of $C_0^\infty(\mathrm{R}^n)$ functions, called the set of test functions in which convergence is defined in the following way : a sequence $\{\phi_\nu\}$ of functions $\phi_\nu \in D$ converges to $\phi \in D$ in D as $\nu \to \nu_0$ if and only if there is a compact set $K \subset \mathrm{R}^n$ such that supp $(\phi_\nu) \subseteq K$ for each ν, supp $(\nu) \subseteq K$ and for every *n*-tiple *k* of nonnegative integers the sequence $\{f^{(k)}\phi_\nu(t)\}$ converges to $f^{(k)}\phi(t)$ uniformly on K as $\nu \to \nu_0$. (see [1,2]).

Distributions (or generalized functions) are objects that generalize the classical notion of functions in mathematical analysis. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense.

Definition 2. A distribution T is continuous linear functional on D. Instead of writing $T(\phi)$, it is conventional to write $<T,\phi>$ for the value of T acting on a test function ϕ. The space of all distributions is denoted by D'.

Schwartz space is the vector space

$$S(\mathrm{R}^n)=\left\{\phi:\mathrm{R}^n \to \mathrm{C},\ \phi \in C^\infty,\ \sup_{x\in\mathrm{R}^n}\left|x^\alpha \partial^\beta \phi(x)\right| < \infty,\ \alpha,\beta \in P^n\right\}.$$

S' is the space of all continuous linear functionals on S , called the space of of tempered distributions.

Proposition 1. A sequence $\{\phi_\nu\}$ of functions $\phi_\nu \in S$ converges to $\phi \in S$ in S as $\nu \to \nu_0$ if and only if

$$\lim_{\nu\to\nu_0}\sup_{x\in\mathbb{R}^n}\left|x^{\alpha} f^{(k)}\left[\phi_\nu(x)-\phi(x)\right]\right| = 0.$$

Let ϕ be an element of one of the above function spaces D or S, and f be a function for which

$$< T_f, \phi >= \int_{\mathbb{R}^n} f(t)\phi(t)dt, \quad \phi\in D(\phi\in S)$$

exists and is finite. Then T_f is regular distribution on D (or S) generated by f (see [3]).

Definition 3. Distributions f and g in $D'(\Omega)$ are equal (in global sense) in open set $\Omega_1 \subset \Omega$ if and only if their restrictions are equal in Ω_1. Distributions f and g are equal (in local sense) in any neighborhood of a point $x \in \Omega$ if and only if are equal in any open neighborhood $U \subset \Omega$ of a point.

Note 1. Fourier transform $\hat{f}$ (see [4]) is continuous linear function from $S(\mathrm{R}^n)$ to $S(\mathrm{R}^n)$.
The (complex) Fourier series for the distribution T is the series

$$T \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \;\text{ with }\; c_n \overset{def}{=} \frac{1}{2\pi} < T, e^{-inx} >, \forall n\in \mathrm{Z}.$$

Proposition 2. A trigonometric series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

converges in D', that is, $\lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{inx}$ exists as a distribution, if and only if there are constants B and β such that

$$\left|c_n\right| \le B\left|n\right|^{\beta}, \forall n \neq 0.$$

For every distribution T the Fourier series converges in S' to T.

Definition 4. The value of distribution f at point x_0 is defined as the limit

$$f(x_0) = \lim_{h\to 0} f(x_0 + hx),$$

if the limit exists in D', that is, if

$$\lim_{h\to 0} < f(x_0+hx), \phi(x) >= f(x_0)\int_{-\infty}^{\infty} \phi(x)dx$$

for each $\phi \in D(\mathrm{R})$.

The Heaviside function is defined by

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

A distribution $f \in D'(\mathrm{R})$ is said to have a distributional jump behavior (or jump behavior) at $x = x_0 \in \mathrm{R}$ if it statisfies the distributional asymptotic relation

$$f(x_0+hx) = c_- H(-x) + c_+ H(x) + o(1),$$

as $h \to 0^+$ in D', $c_{\pm}$ are constants and H is the Heaviside function.

The jump of f at $x = x_0$ is defined as the number $[f]_{x=x_0} = c_+ - c_-$.

A distribution $f \in D'(\mathrm{R})$ is said to have a distributional symmetric jump behavior at $x = x_0 \in$ □ if the jump distribution $\psi_{x_0}(x) = f(x_0+x) - f(x_0-x)$ has a jump behavior at $x=0$. In such a case, we define the jump of f at $x = x_0$ as $[f]_{x=x_0} = [\psi_{x_0}]_{x=0} / 2.$

Note 2. Jump behavior implies symmetric jump behavior, but the converse is not true as shown by *Dirac delta function* (see [4]).

Let us define the subset of upper half-plane $\Delta_\theta^+(x_0)$ as the set of z such that $\theta \leq \arg(z-x_0) \leq \pi - \theta$, where $0 < \theta \leq \pi/2$. Similarly we define the lower half-plane $\Delta_\theta^-(x_0)$.

We may see f as hyperfunction, that is $f(x) = G(x+i0) - G(x-i0)$, where G is analytic for $Imz \neq 0$ or in the sense of distributions it means $f(x) = \lim_{y\to 0^+}(G(x+iy) - G(x-iy))$, then we say G is analytic representation of f.

We say that *U*(*z*), harmonic on $Imz > 0$, is a harmonic representation of $f \in D'($□$)$ if $\lim_{y\to 0^+} U(x+iy) = f(x)$ in $D'(\mathrm{R})$.

Theorem 1. If K is compact subset of Ω, exists $\psi \in C_0^\infty(\Omega)$ with the property $0 \le \psi(x) \le 1$ and $\psi(x) = 1$ in any neighborhood of K.

2 Main Results

Theorem 2. If $f \in D'(\Omega)$ is equal to zero in neighborhood of each point in Ω, then $f = 0$ in Ω.

Proof. Let $\phi \in C_0^\infty(\Omega)$. We take $K = \operatorname{supp}\phi$. Each point of K has neighborhood in which $f = 0$. Since K is compact set, exists finite number of open sets Ω_i, $i = 1, 2, ..., k$, which cover K in which $f = 0$. Now we will show that exists compact sets K_i, $i = 1, 2, ..., k$, such that $K_i \subset \Omega_i$ and $K \subset \bigcup_{i=1}^{k} K_i$.

$C(\bigcup_{i=2}^{k}\Omega_i)$ is closed (as complement of open) and bounded, so it is compact. The set $K \cap C(\bigcup_{i=2}^{k}\Omega_i)$ is a compact (as intersection of finite number of compact sets) and is subset of Ω_1, since $K \subset \bigcup_{i=1}^{k}\Omega_i \Rightarrow C\left(\bigcup_{i=1}^{k}\Omega_i\right) \subset C(K)$ we obtain

$$K \cap C(\bigcup_{i=2}^{k}\Omega_i) = K \cap C(\bigcup_{i=1}^{k}\Omega_i \setminus \Omega_1) = K \cap C(\bigcup_{i=1}^{k}\Omega_i \cap C(\Omega_1)) =$$
$$= \left(K \cap \left(\bigcap_{i=1}^{k} C(\Omega_i)\right)\right) \cup (K \cap \Omega_1) \subset (K \cap C(K)) \cup (K \cap \Omega_1) = \varnothing \cup (K \cap \Omega_1) \subset \Omega_1.$$

Exists bounded open set V_1 such that $\bar{V_1} \subset \Omega$ and $K \cap C(\bigcup_{i=2}^{k}\Omega_i) \subset V_1$.

The sets $V_1, \Omega_2, ..., \Omega_k$ cover K so exists bounded open set V_2 such that $\bar{V_2} \subset \Omega$ and $K \cap C(V_1 \bigcup \bigcup_{i=3}^{k}\Omega_i) \subset V_2$.

In same way we construct sets $V_3, ..., V_k$. If put that $K_i = \bar{V_i}$, $i = 1, ..., k$, obtain compact sets for which is valid $K_i \subset \Omega_i$ and $K \subset \bigcup_{i=1}^{k} K_i$.

From theorem 1, exists a function $\psi_i \in C_0^\infty(\Omega)$ such that $0 \le \psi_i \le 1$ and $\psi_i = 1$ in neighborhood of ψ_i. We put that $\phi_1 = \psi_1;\ \phi_i = \psi_i(1-\psi_1)...(1-\psi_{i-1});\ i = 2,...,k.$ While $\psi_1 + \psi_2(1-\psi_1) = 1-(1-\psi_2)(1-\psi_1)...,$ is obtained

$$\sum_{i=1}^{k} \phi_i = 1-(1-\psi_1)(1-\psi_2)...(1-\psi_k)\,,$$

respectively $\phi_i \ge 0;\ \sum_{i=1}^{k} \phi_i \le 1$ and $\sum_{i=1}^{k} \phi_i = 1$ so exists $\phi \in D(\Omega)$ $\bar{\phi} = \sum_{i=1}^{k} \bar{\phi}\phi_i,$ $\bar{\phi}$ is conjugate function of ϕ.

Next, from the conditions stated in the theorem we have

$$(f,\phi) = \sum_{i=1}^{k} (f,\phi_i\phi) = \sum_{i=1}^{k} (0,\phi_i\phi) = (0,\phi)$$

in the sense of distribution or $f = 0 \quad \text{in} \quad \Omega$.

Theorem 3. If f is a $2\pi-$periodic distribution and has a jump behavior at $x = x_0$, with jump $[f]_{x=x_0}$, then for each positive integer *k* we have that for $0 < \theta \le \pi/2$,

$$\lim_{z\to x_0,\ z\in\Delta_\theta^+(x_0)} (z-x_0)^k \sum_{n=0}^{\infty} n^k c_n e^{inz} = -\frac{(k-1)!}{2\pi(-i)^{k+1}}[f]_{x=x_0}\,.$$

Proof. Firstly we will show that if the condition

$$\lim_{z\to z_0,\ z\in\Delta_\theta^\pm(x_0)} (z-x_0)^k G^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i}[f]_{x=x_0}$$

holds for one analytic representation, then it holds for any analytic representation of f . From *edge of the wedge theorem*, any two such analytic representation differ by an entire function, but for entire function the relation

$$\lim_{z\to z_0,\ z\in\Delta_\theta^\pm(x_0)} (z-x_0)^k G^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i}[f]_{x=x_0}$$

gives 0. We prove that we may assume that $f \in S'$.

Indeed we can decompose $f = f_1 + f_2$ where f_2 is zero in a neighborhood of x_0 and $f_1 \in S'$. Let G_1 and G_2 be analytic representations of f_1 and f_2 , respectively, then G_2 can be continued across a neighborhood of x_0 (once again edge of the wedge theorem), then $G_2(z) = G_2(x_0) + O(|z - x_0|) = O(1)$ as $z \to x_0$. Additionally, f_1 has the same jump behavior as f . Hence we may assume that $f \in S`$.

Let consider the analytic representation

$$G(z) = \begin{cases} \dfrac{1}{2\pi} < \hat{f}_+(t),\, e^{izt} >, & \operatorname{Im} z > 0 \\ -\dfrac{1}{2\pi} < \hat{f}_-(t),\, e^{izt} >, & \operatorname{Im} z < 0 \end{cases}$$

where $f = \hat{f}_- + \hat{f}_+$ is decomposition with supp $\hat{f}_- \subseteq (-\infty, 0]$ and $\hat{f}_+ \subseteq [0, \infty)$.

For the number z on a compact subset of $\Delta_\theta^{\pm}(x_0)$, when we use the relation

$$x^k e^{i\lambda x_0 x} \hat{f}_{\pm}(\lambda x) = (\pm 1)^{k-1} \frac{1}{i\lambda} [f]_{x=x_0} x_{\pm}^{k-1} + o(\frac{1}{\lambda})$$

as $\lambda \to \infty$, we obtain

$$\begin{aligned} G^{(k)}(x_0 + \frac{z}{\lambda}) &= \pm \frac{i^k}{2\pi} \lambda^{k+1} < t^k e^{i\lambda x_0 t} \hat{f}_{\pm}(\lambda t), e^{izt} > \\ &= \pm \frac{(\pm i)^k}{2\pi} [f]_{x=x_0} \lambda^k \int_0^\infty t^{k-1} e^{\pm izt} dt + o(\lambda^k) \\ &= (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0} (\frac{\lambda}{z})^k + o(\lambda^k) \end{aligned}$$

as $\lambda \to \infty$.

Next, we substitute $z = \lambda z - \lambda x_0$, obtain

$$G^{(k)}(x_0 + \frac{\lambda z - \lambda x_0}{\lambda}) = (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0} (\frac{\lambda}{\lambda(z - x_0)})^k + o(\lambda^k)$$

$$(z - x_0)^k G^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0} + o(\lambda^k)$$

equals to

$$\lim_{z \to z_0,\, z \in \Delta_\theta^{\pm}(x_0)} (z - x_0)^k G^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i}[f]_{x=x_0}.$$

While f have jump behavior it implies symmetric jump behavior and every distributionally convergent series may be differentiated term by term (see [5,6,7]), for any positive integer k and $0 < \theta \le \pi / 2$ is valid

$$\lim_{z \to x_0,\, z \in \Delta_\theta^{+}(x_0)} \sum_{n=0}^{\infty} n^k c_n e^{inz} = (-1)^k \frac{(k-1)!}{\pi i^{k+1} (z - x_0)^k}[f]_{x=x_0}$$

we obtain the result

$$\lim_{z \to x_0,\, z \in \Delta_\theta^{+}(x_0)} (z - x_0)^k \sum_{n=0}^{\infty} n^k c_n e^{inz} = -\frac{(k-1)!}{2\pi (-i)^{k+1}}[f]_{x=x_0}.$$

3 Conclusion

In this paper the first result has to do on local equality of two distributions and the second one states if a $2\pi -$periodic distribution has jump behavior at point, then exists the relation of jumps of distributions with Fourier series in a subset of the upper half-plane.

Competing Interests

Authors have declared that no competing interests exist.

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