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On the Solution of Bi-level Large Scale Quadratic Programming Problem with Stochastic Parameters in the Constraints

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Abstract

In this paper, the bi-level large scale quadratic programming problem with stochastic parameters in the constraints (SBLLSQPP) is solved. Randomness is assumed only on the right-hand side of the constraints and the random variables are assumed to be normally distributed. The main features of the proposed solution procedure are based on convert the probabilistic nature of this problem into an equivalent deterministic, Taylor transformation and decomposition algorithms. An illustrative numerical example is given to clarify the main results developed in the paper.

Keywords: Large scale problems, stochastic programming, quadratic programming, bi–level programming.

MSC 2000: 90C06; 90C15; 90C20; 90C99.

1 Introduction

The basic concept of the Bi-level programming (BLP) technique is that a first level decision maker (FLDM) - the leader - sets his goals and/or decisions and then asks each subordinate level of the organization for their optima which are calculated in isolation; the second-level decision

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maker (SLDM) - the follower - decisions are then submitted and modified by the FLDM with consideration of the overall benefit for the organization; and the process continued until an optimal solution is reached. In other words, although the FLDM independently optimizes its own benefits, the decision may be affected by the reaction of the SLDM [1].

Real world optimization problems are often large and nonlinear. A financial optimization problem can have hundreds of thousands of variables and constraints.

Stochastic or probabilistic programming deals with situations where some or all of the parameters of the optimization problem are described by stochastic (or random or probabilistic) variables rather than by deterministic quantities. There may be several sources of random variables depending on the nature and the type of problem.

Many studies are focused on bi-level problem [1,2,3,4]. In [2], Emam proposed an algorithm for solving bi-level integer multi-objective fractional programming problem. In the first phase of the solution algorithm, it began by finding the convex hull of its original set of constraints then simplifying the equivalent problem by transforming it into a separate multi-objective decision-making problem and finally solving the resulting problem by using the ε -constraint method.

Approaches for solving large scale mathematical programs are divided into two classes, direct methods and indirect methods. Direct methods is an existing algorithm such as simplex method in the linear case, which is especially adapted for solving the problem, indirect methods such as decomposition and partitioning methods.

In Osman et al. [5] presented a method for solving a special class of large scale fuzzy multi-objective integer problems depending on the decomposition algorithm. El-Sawy, et.al.[6] introduced an algorithm for decomposing the parametric space in large scale linear vector optimization problems under fuzzy environment.

Benzi et al. [7] developed and compared multilevel algorithms for solving large scale bound constrained nonlinear problems via interior point methods. It shows how a multilevel continuation strategy can be used to obtain good initial guesses for each nonlinear iteration. A minimal surface problem is used to illustrate the various approaches.

Recently, notable studies have been done in the area of stochastic programming problems [8,9,10]. In Pramanik and Banerjee [10] presented an approach to deal with the fuzzy goal programming approach to solve chance constrained quadratic bi-level programming problem. The presented approach converts the chance constraints into equivalent deterministic constraints with prescribed distribution functions and confidence levels. Then a quadratic membership function by using individual best solution based on first order Taylor's series is formed.

In this paper, an attempt to solve bi-level large scale quadratic programming problem with stochastic parameters in the constraints (SBLLSQPP) based on a decomposition algorithm is considered.

This paper is organized as follows: we start in Section 2 by formulating the model of a bi-level large scale quadratic programming problem with stochastic parameters in the constraints. In Section 3, the decomposition method of large scale bi-level linear programming problem is presented. An algorithm for solving a bi-level large scale quadratic programming problem

(BLLSQPP) with stochastic parameters in constraint is suggested in Section 4. In addition, a numerical example is provided in Section 5 to clarify the results and the solution algorithm. Finally, conclusion and future works are reported in Section 6.

2 Problem Formulation and Solution Concept

The bi- level large scale quadratic programming problem [BLLSQPP] with stochastic parameters in the constraints may be formulated as follows:

[First Level]

$$\underset{x_1,x_2}{Max}\, F_1(x) = c_1 x + \frac{1}{2} x^T L_1 x, \tag{1}$$

Where x_3, x_4 solves

[Second Level]

$$\underset{x_3,x_4}{Max} F_2(x) = c_2 x + \frac{1}{2} x^T L_2 x, \tag{2}$$

Subject to

$$x \in G'. \tag{3}$$

Where

$$G' = \{\, pr(a_{01}x_1 + a_{02}x_2 + a_{04}x_4 \le b_0) \ge \alpha_1,$$

$$pr(d_1 x_1 \le b_1) \ge \alpha_2$$

$$pr(d_2 x_2 \le b_2) \ge \alpha_3$$

$$pr(d_4 x_4 \le b_m) \ge \alpha_m,$$

$$x_1, ..., x_4 \ge 0. \}$$

$F_i : R^m \to R, (i = 1,2)$ be the first level objective function, the second level objective function, respectively, (L_1, L_2) are $m \times n$ matrices describing the coefficients of the quadratic terms and (c_1, c_2) are $1 \times m$ matrices, in the above problem (1) – (4), x is m real vector, G is the large scale linear constraint set where, $b = (b_0, ..., b_m)^T$ is $(m+1)$ vector, and

$a_{01},...,a_{04},d_1,...,d_4$ are constants. Therefore, the first level decision maker (FLDM) has x_1,x_2 indicating the first decision level choice and the second level decision maker (SLDM) has x_3,x_4 indicating the second decision level choice. Furthermore p means probability and α_i is a specified probability value.

This means that the linear constraints may be violated some of the time and at most 100(1- α_i) % of the time. For the sake of simplicity, we assume that the random parameters b_i , (i =1, 2,…, m) are distributed normally with known means E{ b_i } and variances V{ b_i }and independently of each other.

Definition 1.

If $x^* \in R^m$ is a feasible solution of the BLLSQPP with probability $\prod_{i=1}^{m}\alpha_i$; no other feasible solution $x \in G$ exists, such that $F_1(x^*) \le F_1(x)$; so x^* is the Pareto optimal solution of the BLLSQPP.

The basic idea in treating problem(SBLLSQPP) is to convert the probabilistic nature of this problem into an equivalent deterministic. In this case, the set of constraints X can be rewritten in the deterministic form as [11]:

$$X' = \left\{ X \in R^n \middle| \sum_{j=1}^{n} a_{ij}x_j \le E(b_i) + K_{\alpha_i}\sqrt{Var(b_i)}, (i = 1,2,....,m), x_j \ge 0, (j = 1,2,...m) \right\} \quad (4)$$

Where K_{α_i} is the standard normal value such that $\Phi(K_{\alpha_i}) = 1 - \alpha_i$; and $\Phi(a)$ represents the "cumulative distribution function" of the standard normal distribution evaluated at a. Thus, problem (BLLSQPP) with stochastic parameters in constraints can be understood as the following deterministic BLLSQPP.

[First Level]

$$\underset{x_1,x_2}{Max}\ f_1(x) = c_1x + \frac{1}{2}x^T L_1 x, \quad (5)$$

Where $x_3,...,x_m$ solves

[Second Level]

$$\underset{x_3,x_4}{Max}\ f_2(x) = c_2x + \frac{1}{2}x^T L_2 x, \quad (6)$$

Subject to

$$x \in G'. \tag{7}$$

Where

$$G' = \{ \quad a_{01}x_1 + a_{02}x_2 + a_{04}x_4 \leq b_0$$

$$d_1 x_1 \leq b_1 ,$$

$$d_2 x_2 \leq b_2 ,$$

$$d_4 x_4 \leq b_m ,$$

$$x_1, \ldots, x_4 \geq 0 \}.$$

3 Decomposition Algorithm for the Bi-level Large Scale Linear Programming Problem

To solve bi-level large scale quadratic programming problem using decomposition algorithm very complex problem, Taylor series can overcome this problem by obtains polynomial objective functions which are equivalent to quadratic objective functions.

$$H_i(x) \cong f_i^{\wedge}(x) = f_i(x_i^*) + \sum_{j=1}^{n} (x_j - x_{ij}^*) \frac{\partial f_i(x_i^*)}{dx_j}, (j = 1,2,3) \tag{8}$$

So the equivalent BLLSLPP can be written as:

[First Level]

$$\underset{x_1, x_2}{Max}\ H_1(x), \tag{9}$$

Where x_3, x_4 solves

[Second Level]

$$\underset{x_3, x_4}{Max}\ H_2(x), \tag{10}$$

Subject to

$$x \in G'. \tag{11}$$

The bi-level large scale linear programming problem is solved by adopting the leader-follower Stakelberg strategy combined with Dantzigand Wolf decomposition method [12].

One first gets the optimal solution that is acceptable to FLDM using the decomposition method to break the large scale problem into n-sub problems that can be solved directly.

The decomposition principle is based on representing the BLLSLPP in terms of the extreme points of the sets $d_j x_j \leq b_j,\ x_j \geq 0,\ j = 1,2,..,m$. To do so, the solution space described by each $d_j x_j \leq b_j, x_j \geq 0, j = 1,2,..,m$ must be bounded and closed.

Then by inserting the FLDM decision variable to the SLDM for him/her to seek the optimal solution using Dantzigand Wolf decomposition method [12], then the decomposition method break the large scale problem into n-sub problems that can be solved directly until the SLDM obtains optimal solution of problem which is the optimal solution to BLLSLPP.

Theorem 1.

The decomposition algorithm terminates in a finite number of iterations, yielding a solution of the large scale problem.

To prove theorem 1 above, the reader is referred to [12].

4 An Algorithm

A solution algorithm to solve Bi-level large scale quadratic programming problem (BLLSQPP) with stochastic parameters in constraint is described in a series of steps. The suggested algorithm can be summarized in the following manner:

Step 1. Determine the means E{ b_i } and Var{ b_i } (i =1, 2,…m).

Step 2. Convert the original set of constraints X of problem (SBLLSQPPs) into the equivalent set of constraints X′.

Step 3.

Formulate the equivalent problem (BLLSQPP).

Step 4.

Convert problem (BLLSQPP) into (BLLSLPP) using Taylor series approach the transformation for the FLDM, SLDM.

Step 5.

Start with the FLDM problem and go to Step 6.

Step 6.

Convert the master problem in terms of extreme points of the sets $d_j x_j \leq b_j, x_j \geq 0$, $j = 1,2,3$.

Step 7.

Determine the extreme points $x_j = \sum_{k=1}^{k_j} \beta_{jk} \hat{x}_{jk.}, j = 1,2,3$ using Balinski's algorithm [13].

Step 8.

Set $k = 1$.

Step 9.

Compute $z_{jk} - c_{jk} = C_B B^{-1} P_{jk} - c_{jk}$, go to Step 10.

Step 10.

If $\overset{*}{z}_{jk} - \overset{*}{c}_{jk} \le 0$, then go to Step 11; otherwise, the optimal solution has been reached, go to Step 16.

Step 11.

Determine $\hat{X}_{jk}$ associated with min{ $\overset{*}{z}_{jk} - \overset{*}{c}_{jk}$ }, go to step 12.

Step 12.

B_{jk} associated with extreme point $\hat{X}_{jk}$ must enter the solution, go to step 13.

Step 13.

Determine leaves variable, go to step 14.

Step 14.

The new basis is determined by replacing the vector associated with leaving variable with the vector B_{jk}, go to step 15.

Step 15.

Set $k = k + 1$, go to Step9.

Step 16.

If the SLDM obtain the optimal solution go to Step 19, otherwise go to Step 17.

Step 17.

Set $(x_1, x_2) = (x_1^F, x_2^F)$ to the SLDM constraints, go to Step 18.

Step 18.

The SLDM formulate his problem, go to Step 8.

Step 19.

$\left(x_1^F, x_2^F, x_3^S, x_4^S\right)$ is as an optimal solution for bi-level large scale linear programming problem, then stop .

5 Numerical Example

To demonstrate the solution for (SBLLSQPP), let us consider the following problem:

[First Level]

$$\underset{x_1, x_2}{Max}\ F_1(x_1, x_2) = \underset{x_1, x_2}{Max} 2x_1^2 + 3x_2^2 + x_4$$

Where x_3, x_4 solves

[Second Level]

$$\underset{x_3,x_4}{Max} F_2(x_3,x_4) = \underset{x_3,x_4}{Max} x_1 + 3x_3^2 + 4x_4^2$$

Subject to

$$pr(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le v_1) \ge 0.0228,$$

$$pr(4x_1 + x_2 \le v_2) \ge 0.0668,$$

$$pr(4x_3 + 2x_4 \le v_3) \ge 0.0287,$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

where v_i *(i =1,2,3)* has independent normal distribution with the following means and variances:

Random variable	v_1	v_2	v_3
Mean	10	8	5
Variance	225	256	100

From standard normal tables, we have:

K α_1 =2, K α_2 =1.5, K α_3 =1.9.

Now the (SBLLSQPP) can be understood as the following deterministic bi-level large scale quadratic programming problem (BLLSQPP):

[First Level]

$$\underset{x_1,x_2}{Max}\ F_1(x_1,x_2) = \underset{x_1,x_2}{Max} 2x_1^2 + 3x_2^2 + x_4$$

Where x_3, x_4 solves

[Second Level]

$$\underset{x_3,x_4}{Max} F_2(x_3,x_4) = \underset{x_3,x_4}{Max} x_1 + 3x_3^2 + 4x_4^2$$

Subject to

$$x_1 + x_2 + x_3 + x_4 \le 40,$$

$$4x_1 + x_2 \le 32,$$

$$4x_3 + 2x_4 \le 24,$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

Using the 1^{st} order Taylor polynomial series to convert the quadratic function to linear function. Therefore, the (BLLSQPP) is written as:

[First Level]

$$\underset{x_1,x_2}{Max}\, F_1(x_1,x_2) = \underset{x_1,x_2}{Max} 4x_1 + 6x_2 + x_4 - 2$$

Where x_3, x_4 solves

[Second Level]

$$\underset{x_3,x_4}{Max}\, F_2(x_3,x_4) = \underset{x_3,x_4}{Max}\, x_1 + 6x_3 + 8x_4 - 7_{20}$$

Subject to

$$x_1 + x_2 + x_3 + x_4 \le 40,$$

$$4x_1 + x_2 \le 32,$$

$$4x_3 + 2x_4 \le 24,$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

The FLDM problem formulation is as follows:

$$Max\ F_1(x) = \underset{x_1,x_2}{Max} 4x_1 + 6x_2 + x_4 - 2,$$

Subject to

$$x_1 + x_2 + x_3 + x_4 \le 40,$$

$$4x_1 + x_2 \le 32,$$

$$4x_3 + 2x_4 \le 24,$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

After 3 iterations the FLDM obtain his optimal solution.

$$\left(x_1^{\;F}, x_2^{\;F}, x_3^{\;F}, x_4^{\;F}\right) = (0,32,6,0).$$

Now set $(x_1, x_2) = (0,32)$ to the SLDM constraints.

Secondly, the SLDM solves his/her problem as follows:

$$\underset{x_3,x_4}{Max}\, F_2\left(x_3, x_4\right) = \underset{x_3,x_4}{Max}\, 6x_3 + 8x_4 + 25,$$

Subject to

$$x_3 + x_4 \le 8,$$

$$4x_3 + 2x_4 \le 24,$$

$$x_3, x_4 \ge 0.$$

The SLDM perform the same action like FLDM till he obtains optimal solution $\left(x_3^{\;S}, x_4^{\;S}\right) = (0,8).$

So $\left(x_1^{\;F}, x_2^{\;F}, x_3^{\;S}, x_4^{\;S}\right) = (0,32,0,8)$ is the optimal solution for bi-level large scale linear programming problem, where $F_1 = 198, F_2 = 89$.

6 Conclusion

In this paper, a powerful approach was based on decomposition algorithm to solve bi-level large scale quadratic programming problem with stochastic parameters in the constraints (BLLSQPP). We assumed that there is randomness in the right-hand sides of the constraints only and that the random variables are normally distributed. The basic idea in treating problem (SBLLSQPP) was to convert the probabilistic nature of this problem into an equivalent deterministic. Then Taylor series was combined with a decomposition algorithm to obtain the optimal solution of this problem.

Certainly, there are many other points in this area of large scale multi-level optimization that should be studied. One may have to consider the following open points for future research:

1- Multi-level multi-objective large scale fractional programming problem with stochastic parameters in both objective functions and constraints.
2- Multi -level multi-objective large scale fractional programming problem with fuzzy parameters in the objective functions and in the constraints and with integrality conditions.
3- Multi- level large scale fractional programming problem with rough parameters in the objective functions and in the constraints and with integrality conditions.

Competing Interests

Authors have declared that no competing interests exist.

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