Hybrid Numerical Method with Block Extension for Direct Solution of Third Order Ordinary Differential Equations

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Abstract

This paper focuses on the development of a hybrid method with block extension for direct solution of initial value problems (IVPs) of general third-order ordinary differential equations. Power series was used as the basis function for the solution of the IVP. An approximate solution from the basis function was interpolated at some selected off-grid points while the third derivative of the approximate solution was collocated at all grid and off-grid points to generate a system of linear equations for the determination of the unknown parameters. The derived method was tested for consistency, zero stability, convergence and absolute stability. The method was implemented with five test problems including the Genesio equation to confirm its accuracy and usability. The rate of convergence (ROC) reveals that the method is consistent with the theoretical order of the proposed method. Comparison of the results with some existing methods shows the superiority of the accuracy of the method.

Keywords

Hybrid, Modified Block, Grid Points, Interpolation and Collocation

1. Introduction

The focus of this article is to find an approximate solution on a given interval to third order initial value problems (IVP) of the type

$$y'''(x) = f(x, y(x), y'(x)), y(x) = \alpha_a, y'(x) = \alpha_b, y''(x) = \alpha_c \tag{1}$$

where $x \in [a,b] \subset \mathbb{R}$ and $y(x), f(x, y(x), y'(x), y''(x)) \in \mathbb{R}^n$. In recent time, direct numerical solution of (1) without reduction to equivalent first-order initial value problems (see:[1] [2] [3] [4] [5]) has become subject of research by

several authors. This method was extensively discussed in ([6] [7] [8] [9] [10]) to mention but a few, they developed Linear Multistep Method (LMM) which mode of implementation is Predictor-Corrector form for the solution of initial value problems of ordinary differential equations of the type (1). As reported by [10] [11], the major drawback of this approach of implementation is that the methods are not self-starting and thus required the development of predictors which are usually of lower order, hence reducing the accuracy of the methods.

Recently, in order to remove the difficulties usually encountered by adopting this mode of solution, researchers ([1] [11]-[22]) have proposed direct methods other than Predictor-Corrector methods whose modes of implementation are in block-by-block manner which was first introduced by Milne [23] as a starting step for predictor-corrector. This monumental success has greatly removed the burden of developing predictors and hence resulted in methods of uniform orders that yielded more accurate results. The block-by-block technique has also made it easier to handle the general type of (1) which has been a major concern in the past years.

The quest for numerical methods with better accuracy has also led to the introduction of hybrid linear multistep methods which have recorded high success since its introduction. These successes motivated us to propose a hybrid method with block extension for the solution of (1).

In the next section, we discuss in detail the derivation of the proposed method with its implementation in block mode, followed by analysis of the proposed method to establish the numerical stability, numerical example to demonstrate the efficiency advantages of the proposed method and subsequently. Conclusion was drawn on the performance of the proposed method when applied to solve the numerical examples.

2. Mathematical Formulation

In order to obtain a numerical formula for the approximate solution of (1), the function

$$y(x) = \sum_{v}^{(c+i)-1} g_v x^v \tag{2}$$

is considered as the basis where x is continuous within the interval $[a,b]$, c and i denote collocation and interpolation points respectively. Variable g_v's are coefficients to be determined. The third derivative of (2) equated to (1) is given as

$$\sum_{v}^{(c+i)-1} v(v-1)(v-2) g_v x^{v-3} = f(x, y, y', y'') \tag{3}$$

Evaluating (2) at $x = x_{n+v}, v = \dfrac{2}{8}, \dfrac{4}{8}, \dfrac{6}{8}$, (3) at $x = x_{n+v}, v = 0\left(\dfrac{2}{8}\right)1$ using

$\tau = \dfrac{x - x_{n+k}}{h}$ yield the following interpolation and collocation matrix

$$XA = B \qquad (4)$$

where

$$A = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix}, \quad B = \begin{pmatrix} y_{n+\frac{2}{8}} \\ y_{n+\frac{4}{8}} \\ y_{n+\frac{6}{8}} \\ f_n \\ f_{n+\frac{2}{8}} \\ f_{n+\frac{4}{8}} \\ f_{n+\frac{6}{8}} \\ f_{n+1} \end{pmatrix},$$

$$X = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{1024} & \frac{1}{4096} & \frac{1}{16384} \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} \\ 1 & \frac{3}{4} & \frac{9}{16} & \frac{27}{64} & \frac{81}{256} & \frac{243}{1024} & \frac{729}{4096} & \frac{2187}{16384} \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 6 & \frac{15}{4} & \frac{15}{8} & \frac{105}{128} \\ 0 & 0 & 0 & 6 & 12 & 15 & 15 & \frac{105}{8} \\ 0 & 0 & 0 & 6 & 18 & \frac{135}{4} & \frac{405}{8} & \frac{8505}{128} \\ 0 & 0 & 0 & 6 & 24 & 60 & 120 & 210 \end{pmatrix}.$$

where $f_{n+v} = f\left(x_{n+v}, y'_{n+v}, y''_{n+v}\right)$, $y_{n+v} \approx y\left(x_{n+v}\right)$. Solving the matrix Equation (4) for coefficients g_v's and substituting into (2) yields after some simplification the continuous method

$$\bar{y}(x) = \zeta_{\frac{1}{4}} y_{n+\frac{1}{4}} + \zeta_{\frac{1}{2}} y_{n+\frac{1}{2}} + \zeta_{\frac{3}{4}} y_{n+\frac{3}{4}} + h^3 \left(\Theta_0(x) f_n + \Theta_{\frac{1}{4}}(x) f_{n+\frac{1}{4}} \right.$$
$$\left. + \Theta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} + \Theta_{\frac{3}{4}}(x) f_{n+\frac{3}{4}} + \Theta_1(x) f_{n+1} \right) \qquad (5)$$

with the following coefficients:

$$\zeta_{\frac{1}{4}} = 8\tau^2 - 10\tau + 3$$

$$\zeta_{\frac{1}{2}} = -16\tau^2 + 16\tau - 3$$

$$\zeta_{\frac{3}{4}} = 8\tau^2 - 6\tau + 1$$

$$\Theta_0 = \frac{1}{322560}(4\tau - 1)(2\tau - 1)(4\tau - 3)\left(512\tau^4 - 1472\tau^3 + 1360\tau^2 - 400\tau + 7\right)$$

$$\Theta_{\frac{1}{4}} = -\frac{1}{80640}(4\tau-3)(2\tau-1)(4\tau-1)\left(512\tau^4 - 1248\tau^3 + 688\tau^2 + 258\tau - 203\right)$$

$$\Theta_{\frac{1}{2}} = \frac{1}{53760}(4\tau-3)(2\tau-1)(4\tau-1)\left(512\tau^4 - 1024\tau^3 + 240\tau^2 + 272\tau + 147\right)$$

$$\Theta_{\frac{3}{4}} = -\frac{1}{80640}(4\tau-3)(2\tau-1)(4\tau-1)\left(512\tau^4 - 800\tau^3 + 16\tau^2 + 62\tau + 7\right)$$

$$\Theta_1 = \frac{1}{322560}(4\tau-1)(2\tau-1)(4\tau-3)\left(512\tau^4 - 576\tau^3 + 16\tau^2 + 48\tau + 7\right)$$

Evaluating the continuous scheme (5) at $\tau = 0,1$ and its first and second derivatives at $\tau = 0$ yield two discrete, one first and second derivatives schemes. These can be represented in a block matrix finite difference form as

$$\Upsilon^0 Y_{m,0} = \Upsilon^1 Y_{m,1} + h\Upsilon^2 Y_{m,2} + h^2\Upsilon^3 Y_{m,3} + h^3\bar{H}F_{m,0} + h^3 H F_{m,1} \tag{6}$$

where T denotes the transpose,

$$H = \begin{pmatrix} \dfrac{107}{64512} & -\dfrac{103}{107520} & \dfrac{43}{107520} & -\dfrac{47}{645120} \\ \dfrac{83}{5040} & -\dfrac{1}{168} & \dfrac{13}{5040} & -\dfrac{19}{40320} \\ \dfrac{1863}{35840} & -\dfrac{243}{35840} & \dfrac{45}{7168} & -\dfrac{81}{71680} \\ \dfrac{34}{315} & \dfrac{1}{210} & \dfrac{2}{105} & -\dfrac{1}{504} \end{pmatrix};$$

$$\bar{H} = \begin{pmatrix} 0 & 0 & 0 & \dfrac{113}{71680} \\ 0 & 0 & 0 & \dfrac{331}{40320} \\ 0 & 0 & 0 & \dfrac{331}{40320} \\ 0 & 0 & 0 & \dfrac{31}{840} \end{pmatrix}; \quad \Upsilon^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\Upsilon^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \Upsilon^2 = \begin{pmatrix} 0 & 0 & 0 & \dfrac{1}{4} \\ 0 & 0 & 0 & \dfrac{1}{2} \\ 0 & 0 & 0 & \dfrac{3}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \Upsilon^3 = \begin{pmatrix} 0 & 0 & 0 & \dfrac{1}{32} \\ 0 & 0 & 0 & \dfrac{1}{8} \\ 0 & 0 & 0 & \dfrac{9}{32} \\ 0 & 0 & 0 & \dfrac{1}{2} \end{pmatrix}$$

Vectors $Y_{m,0}, Y_{m,1}, Y_{m,2}, Y_{m,3}, F_{m,0}$ and $F_{m,1}$ are

$$Y_{m,0} = \left(y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}, y_{n+\frac{3}{4}}, y_{n+1}\right)^{\mathrm{T}}, \quad Y_{m,1} = \left(y_{n-\frac{1}{4}}, y_{n-\frac{1}{2}}, y_{n-\frac{3}{4}}, y_n\right)^{\mathrm{T}},$$

$$Y_{m,2} = \left(y_{n-\frac{1}{4}}, y_{n-\frac{1}{2}}, y_{n-\frac{3}{4}}, y_n'\right)^{\mathrm{T}}, \quad Y_{m,3} = \left(y_{n-\frac{1}{4}}, y_{n-\frac{1}{2}}, y_{n-\frac{3}{4}}, y_n''\right)^{\mathrm{T}},$$

$$F_{m,0} = \left(f_{n-\frac{1}{4}}, f_{n-\frac{1}{2}}, f_{n-\frac{3}{4}}, f_n\right)^{\mathrm{T}}, \quad F_{m,1} = \left(f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1}\right)^{\mathrm{T}}.$$

3. Analysis of the Proposed Method

This section presents analysis of the proposed method (6) vis-a-vis the order, consistency, zero-stability and convergence.

The linear operator associated with the block method (6) is

$$L[y(x);h] = \Upsilon^0 Y_{m,0} - \left(\Upsilon^1 Y_{m,1} + h\Upsilon^2 Y_{m,2} + h^2 \Upsilon^3 Y_{m,3} + h^3 \overline{HF}_{m,0} + h^3 HF_{m,1} \right) \quad (7)$$

where $y(x)$ is an arbitrary function which is continuously differentiable on $[a,b]$. Following Lambert [5] and Fatunla [20], the term in (6) can be written as a Taylor series expansion about the point x to obtain the expression,

$$L[y(x);h] = \overline{c}_0 y(x) + \overline{c}_1 h y'(x) + \overline{c}_2 h^2 y''(x) + \cdots + \overline{c}_p h^p y^p(x) + \cdots, \quad (8)$$

where the constant coefficients $\overline{c}_p, p = 0,1,2,\cdots$ are given as follows:

$$\overline{c}_0 = \zeta_k + \zeta_u + \zeta_v + \zeta_w$$

$$\overline{c}_1 = k\zeta_k + u\zeta_u + v\zeta_v + w\zeta_w$$

$$\overline{c}_2 = \frac{1}{2!}\left(k^2\zeta_k + u^2\zeta_u + v^2\zeta_v + w^2\zeta_w \right)$$

$$\overline{c}_3 = \frac{1}{3!}\left(k^3\zeta_k + u^3\zeta_u + v^3\zeta_v + w^3\zeta_w \right) - \frac{1}{0!}\left(\Theta_0 + \Theta_k + \Theta_u + \Theta_v + \Theta_w \right)$$

$$\overline{c}_4 = \frac{1}{p!}\left(k^4\zeta_k + u^4\zeta_u + v^4\zeta_v + w^4\zeta_w \right) - \frac{1}{(p-3)!}\left(k\Theta_k + u\Theta_u + v\Theta_v + w\Theta_w \right)$$

$$\vdots$$

$$\overline{c}_p = \frac{1}{p!}\left(k^p\zeta_k + u^p\zeta_u + v^p\zeta_v + w^p\zeta_w \right)$$

$$- \frac{1}{(p-3)!}\left(k^{p-3}\Theta_k + u^{p-3}\Theta_u + v^{p-3}\Theta_v + w^{p-3}\Theta_w \right), p = 4,5,\cdots$$

Going by Lambert [5], the mutistep collocation method (6) has order p if

$$L[y(x);h] = 0\left(h^{p+1}\right), \quad \overline{c}_0 = \overline{c}_1 = \cdots = \overline{c}_p = 0, \quad \overline{c}_{p+3} \neq 0$$

Therefore $\overline{c}_{p+3}$ is the error constant and $\overline{c}_{p+3} h^{p+3} y^{p+3}(x_n)$ is the principal local truncation error at point x_n. The order of the proposed method (6) and the corresponding error constant are as reported in Table 1.

Definition 1 (consistency).

The proposed method (6) is said to be consistent if the order of method is greater than or equal to one, that is if $p \geq 1$. In addition to

1) $\rho(1) = 0$ and

Table 1. Order of accuracy and error constant of the proposed method.

Scheme		Order	Error constant
	$y_{n+\frac{1}{4}}$	5	$1.55E-06$
(6)	$y_{n+\frac{1}{2}}$	5	$3.39E-07$
	$y_{n+\frac{3}{4}}$	5	$5.26E-08$
	y_{n+1}	5	$8.28E-07$

2) $\rho'(1) = \sigma(1)$ where $\rho(z)$ and $\sigma(z)$ are 1st and 2nd characteristics polynomial respectively.

Definition 2 (Zero-stability).

The block method (6) is said to be zero-stable if the roots

$$\rho(z) = \det\left[\sum_{i=0}^{k} \Upsilon^{(i)} z^{k-i}\right] \quad (9)$$

satisfies $|z_i| \leq 1, i = 1, \cdots, k$ and the roots with $|z_i| = 1$, the multiplicity must not exceed one. Applying (9) to the proposed method (6) yields the following

$$\rho(z) = \left\| \begin{bmatrix} z & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} z & 0 & 0 & -1 \\ 0 & z & 0 & -1 \\ 0 & 0 & z & -1 \\ 0 & 0 & 0 & z-1 \end{bmatrix} \right\| = z^3(z-1) \quad (10)$$

This result shows that the method is zero-stable.

Definition 3 (convergence).

The necessary and sufficient condition for the proposed method (6) to be convergent are that it must be consistent and zero-stable according to Dahlquist see [5]. Hence, by definitions 4 and 5 the method is convergent.

Stability Domain of the Proposed Method

In order to study the stability domain of the proposed method (6), the test equations

$$y' = \lambda y \quad (11)$$

$$y'' = \lambda^2 y \quad (12)$$

and

$$y''' = \lambda^3 y \quad (13)$$

are applied to the block method (6) with $z = \lambda y$ and η represents the roots of the first characteristic polynomial of the block method (6). This is then reformulated as a general linear method as discussed in [24]. The partition $(s_1 + s_2) \times (s_1 + s_2)$ matrix is expressed in the form

$$\begin{bmatrix} Y \\ \vdots \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & V \\ \vdots & \ddots & \vdots \\ B & & U \end{bmatrix} \begin{bmatrix} h^3 f(y) \\ \vdots \\ Y_{i-1} \end{bmatrix} \quad (14)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \dfrac{113}{71680} & \dfrac{107}{64512} & -\dfrac{103}{107520} & \dfrac{43}{107520} & \dfrac{47}{645120} \\ \dfrac{331}{40320} & \dfrac{83}{5040} & -\dfrac{1}{168} & \dfrac{13}{5040} & -\dfrac{19}{40320} \\ \dfrac{331}{40320} & \dfrac{1863}{35840} & -\dfrac{243}{35840} & \dfrac{45}{7168} & \dfrac{81}{71680} \\ \dfrac{31}{840} & \dfrac{34}{315} & \dfrac{1}{210} & \dfrac{2}{105} & -\dfrac{1}{504} \end{bmatrix},$$

$$B = \begin{bmatrix} \dfrac{113}{71680} & \dfrac{107}{64512} & -\dfrac{103}{107520} & \dfrac{43}{107520} & -\dfrac{47}{645120} \\ \dfrac{31}{840} & \dfrac{34}{315} & \dfrac{1}{210} & \dfrac{2}{105} & -\dfrac{1}{504} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}, f(y) = \begin{bmatrix} f_n & f_{n+\frac{1}{4}} & f_{n+\frac{1}{2}} & f_{n+\frac{3}{4}} & f_{n+1} \end{bmatrix}^{\mathrm{T}},$$

$$Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+1} \end{bmatrix}, Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_n \end{bmatrix}$$

By solving the stability function

$$p(\eta, z) = \left| \eta I - \left(v + z B_m (I - z A_m)^{-1} U_m \right) \right| \qquad (15)$$

yields the polynomial

$$p(\eta, z) = \frac{\eta}{315} \begin{bmatrix} \dfrac{\begin{array}{l} 2835\eta z^4 + 4737600\eta z^3 + 50591 z^4 + 417312000\eta z^2 \\ -134158720 z^3 - 145871596800 z^2 + 124853944320000\eta \\ -20808990720000 z - 124853944320000 \end{array}}{9z^4 + 15040 z^3 + 1324800 z^2 + 396361728000} \end{bmatrix} \qquad (16)$$

(16) and its derivatives are then plotted in the MATLAB environment given the stability region displayed in **Figure 1**.

Definition 4 (Lambert and Watson [25]).

Method (6) is P-stable if its interval of periodicity is $(0, \infty)$. It is clearly
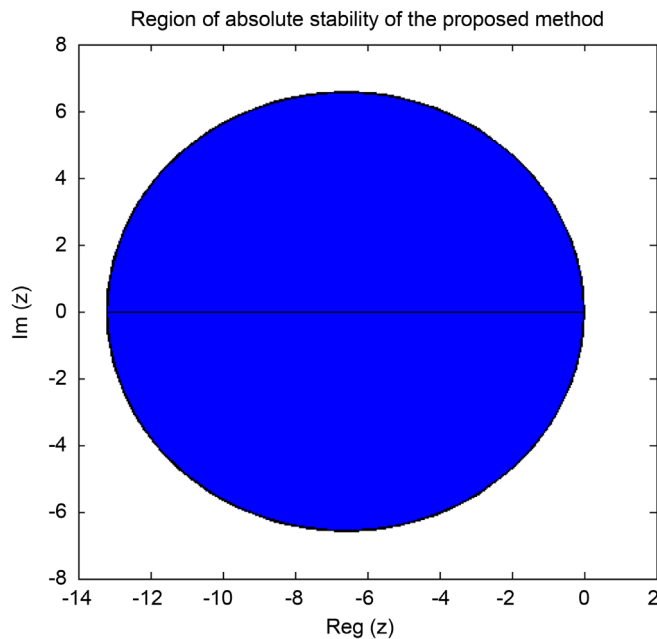
Figure 1. This figure depicts the region of absolute stability of the proposed method generated by plotting Equation (16).

shown in **Figure 1** that the block method (6) is *P*-stable.

4. Numerical Example

Example I

The first numerical example to be considered is the oscillatory problem

$$y''' = 3\sin x, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad h = 0.1$$

with the theoretical solution

$$y(x) = 3\cos x + \frac{x^2}{2} - 2.$$

This example was solved by [11] [15] [18] [26]. The numerical solution, exact solution and absolute error generated by the proposed method when applied to example I are as presented in **Table 2**. The last column of the Table shows the errors generated by method in [26] when applied to example I. It is obvious from the table that the proposed method is better in term of accuracy when compared with the method in [26].

Example II

The second example considered is the special third order problem

$$y''' = e^x, \quad y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 5, \quad h = 0.1$$

with the theoretical solution

$$y(x) = y(x) = 2 + 2x^2 + e^x$$

Source: [26]. The solution curve is shown in **Figure 2**.

Example III

Another example considered is a general third order problem

$$y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22,$$
$$y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -12, \quad h = 0.1$$

with the theoretical solution

Table 2. Results of Example I solved with the proposed method.

X-value	Exact Result	Computed	Errors	Error [26]
0.1000	0.990012495834077020	0.9900124958340308421 25648	4.645616E(−14)	1.743050e−14
0.2000	0.960199733523725120	0.9601997335235394554 37233	1.854379E(−13)	1.082467e−13
0.3000	0.911009467376818090	0.9110094673764022698 66494	4.157891E(−13)	2.711165e−13
0.4000	0.843182982008654940	0.8431829820079196531 22556	7.355953E(−13)	5.079270e−13
0.5000	0.757747685671117830	0.7577476856699759449 13026	1.142203E(−12)	8.164580e−13
0.6000	0.656006844729034370	0.6560068447290348917 2286	1.632248E(−12)	1.199707e−12
0.7000	0.539526561853464590	0.5395265618512635939 01655	2.201685E(−12)	1.654343e−12
0.8000	0.410120128041495670	0.4101201280386504314 37839	2.845831E(−12)	1.674639e−10
0.9000	0.269829904811992540	0.2698299048084339561 43307	3.559413E(−12)	3.336392e−10
1.0000	0.120906917604417960	0.1209069176000825343 34178	4.336618E(−12)	5.001723e−10
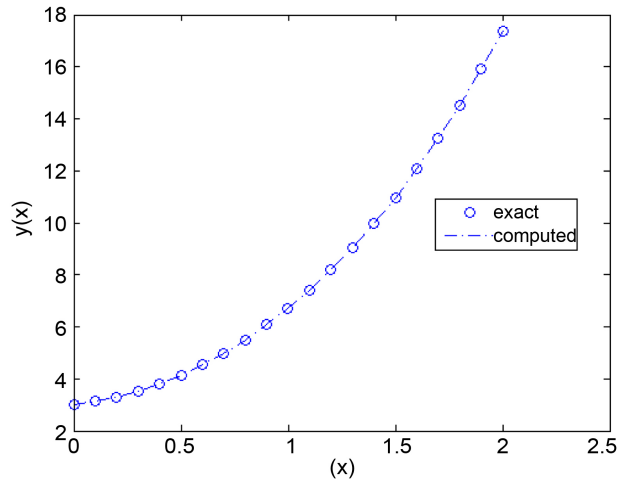
Figure 2. Solution curve obtained by our method and the exact solution of example II.

$$y(x) = -2e^{3x} + e^{-2x} + x^2 - 1.$$

The theoretical solution at $x = 1$ is $y(1) \approx -40.0357385631387227899630$. The errors were obtained at $x = 1$ using our method at a fixed step-size $h = 0.1; 0.05; 0.025; 0.0125; 0.00625$. The numerical results are compared with those of [27]. For this example, the maximum error was compared with those reported in [18] in **Table 3** for $h = 0.01$ and it was observed that our method perform better. The ROC, computed solutions and maximum error of the proposed method are reported in **Table 4**. The Table also shows the performance of our method as compared with method in [27].

Example IV

General nonlinear third order equation

$$y''' = y'(2xy'' + y'), \quad y(0) = 1, \quad y'(0) = 1/2, \quad y''(0) = 0.1, \quad h = 0.1$$

with the theoretical solution

$$y(x) = 1 + \frac{1}{2}\log\left(\frac{2+x}{2-x}\right)$$

is also considered. Source: [28]. **Figure 3** is the graph of the solution of this problem.

Application to solve nonlinear Genesio equation

The chaotic Genesio equation reported in [17] given as

$$y''' = -\alpha y'' - \beta y' + f(y(x))$$

with

$$f(y(x)) = -\gamma y(x) + y^2(x)$$

$$y(0) = 0.2, \quad y'(0) = -0.3, \quad y''(0) = 0.1, \quad x \in [a, b]$$

where $\alpha = 1.2, \beta = 2.92$ and $\gamma = 6$ are the positive constants that satisfied

$$\alpha\beta < \gamma$$

Table 3. Results of Example III solved with the proposed method.

b	h	Method	Step	Maximum Error
		Proposed method	30	9.6015e(−17)
		BHCM	34	7.48e(−17)
	0.01	Adams	100	6.40e(−10)
		Olabode	34	8.89e(−13)
		Adesanya	25	1.75e(−14)

Table 4. Results of Example III solved with the proposed method.

h	y	Max. Error	Error in [27]	ROC
0.1	−40.0357384989252357390316	$6.421349E(-8)$	1:340886(−03)	-
0.05	−40.0357385621393199701503	$9.994028E(-10)$	9:258900(−05)	6.00
0.025	−40.0357385631231226524929	$1.560014E(-11)$	6.075364(−06)	6.00
0.0125	−40.0357385631384791804478	$2.436095E(-13)$	3.889526(−07)	6.00
0.00625	−40.0357385631387191829967	$3.606966E(-15)$	2.460220(−08)	6.08
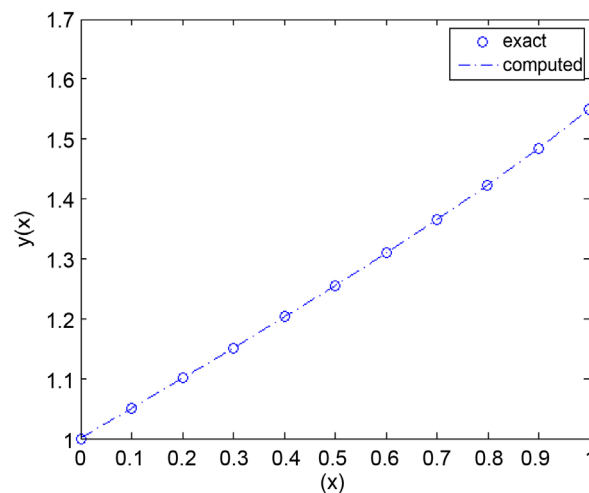
Figure 3. Solution curve of example IV obtained by the proposed method.

for the solution to exist. The solution of this problem is presented in **Figure 4(a)** and **Figure 4(b)** in the intervals $[0,10]$ and $[0,100]$ respectively.

5. Conclusion

In this work, hybrid method with block extension for the direct solution of third order ordinary differential equations has been proposed. Numerical examples are considered to demonstrate the efficiency advantage of the method especially the Genesio equation which is chaotic in nature. The analysis, stability and numerical examples revealed that the proposed method is efficient for direct solution of third order ordinary differential equations.
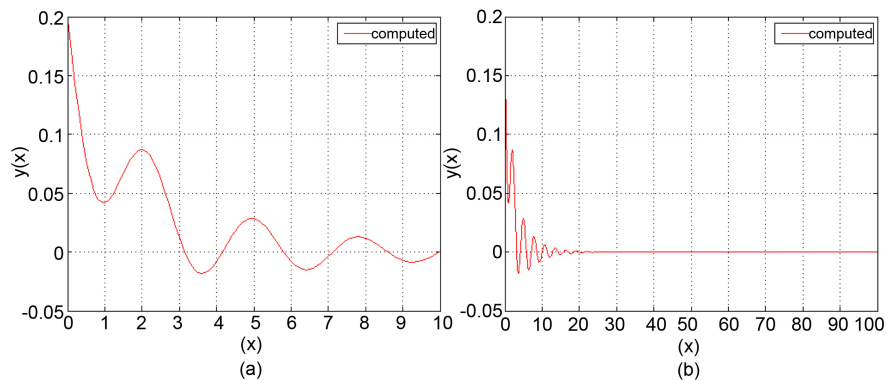
Figure 4. Solution curve of the Genesio equation for $\alpha = 1.2, \beta = 2.92$ and $\gamma = 6$ for step size $h = 0.1$ within [0, N]: (a) $N = 10$ and (b) $N = 100$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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