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A New Iterative Scheme for Common Solution of Equilibrium Problems, Variational Inequalities and Fixed Point of k-strictly Pseudo-contractive Mappings in Hilbert Spaces

Cyril Dennis Enyi[*1] **and Olaniyi Samuel Iyiola**[1]

[1] *Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Saudi Arabia.*

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Abstract

In this paper, we present a new iterative scheme for finding a common point among the set of solution of equilibrium problems, the set of solution to a variational inequality problem and the fixed point set of $\tilde{k}$-strictly pseudo-contractive mappings in a real Hilbert space. We then prove that the proposed scheme converges strongly to a common element which is the solution of a variational inequality problem, system of equilibrium problems, and a fixed point of $\tilde{k}$-strictly pseudo-contractive mappings. These results improve and generalize recent works in this direction.

Keywords: Fixed point; Nonexpansive mapping; $\tilde{k}$-strictly pseudo-contractive mappings; System of equilibrium problems; Variational inequality problem.

2010 Mathematics Subject Classification: 47H10

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H and $\{F_k\}_{k\in\Delta}$ a countable family of bifunctions from $C \times C$ to $\mathbb{R}$. The equilibrium problem associated with the family $\{F_k\}_{k\in\Delta}$ where Δ is an arbitrary index set, is to find $x \in C$ such that

$$F_k(x, y) \geq 0, \forall k \in \Delta, \ \forall y \in C. \tag{1.1}$$

Assume Δ is singleton, we have that (1.1) becomes the equilibrium problem of finding $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \tag{1.2}$$

**Corresponding author: E-mail: cenyi@kfupm.edu.sa*

We denote the set of solutions of (1.2) by EP(F).
Combettes and Hirstoaga [1] in 2005, proved a strong convergence theorem for an iterative scheme for finding the best approximation to the initial data when EP(F) is nonempty, Korpelevich [2].

Given a map $T : C \to H$, let $F(x, z) = \langle Tx, z - x\rangle,\ \forall x, z \in C$. Therefore problem (1.2) becomes a variational inequality problem of finding $x \in C$ such that

$$F(x, z) = \langle Tx, z - x\rangle \geq 0,\ \forall z \in C \tag{1.3}$$

The set of solution of (1.3) is denoted by VI(C, A).

Problem (1.1) is general since numerous problems in optimization, physics, economics, variational inequalities and minimax problems are special cases; see ([3],[4],[5]).

Definition 1.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. A map $T : C \to H$ is said to be nonexpansive if for all $x, z \in C$ we have*

$$\|Tx - Tz\| \leq \|x - z\|.$$

We denote the fixed point set of T by Fix(T).

Definition 1.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. A map $T : C \to H$ is said to be k-strictly pseudo-contractive if there exists a constant $0 \leq k < 1$ such that for all $x, z \in C$*

$$\|Tx - Tz\|^2 \leq \|x - z\|^2 + k\|(I - T)x - (I - T)z\|^2. \tag{1.4}$$

In a real Hilbert space it follows that (1.4) is equivalent to

$$\langle Tx - Tz, x - z\rangle \leq \|x - z\|^2 - \frac{1 - k}{2}\|(I - T)x - (I - T)z\|^2. \tag{1.5}$$

Definition 1.3. *For any $x \in H$, we define the map $P_C : H \to C$ satisfying*

$$\|x - P_C x\| \leq \|x - z\| \ \ \forall z \in C$$

P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - z\rangle \geq 0 \ \ \forall x \in H \text{ and } \forall z \in C. \tag{1.6}$$

Clearly (1.6) is equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2 \ \ \forall x \in H \text{ and } \forall z \in C. \tag{1.7}$$

Definition 1.4. *A mapping A of C into H is called monotone if*

$$\langle Ax - Az, x - z\rangle \geq 0 \ \ \forall x, z \in C,$$

A is called α-inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - z, Ax - Az\rangle \geq \alpha\|Ax - Az\|^2 \ \ \forall x, z \in C,$$

also A is L-Lipschitz-continuous if there exists $L > 0$ such that for all $x, z \in C$

$$\|Ax - Az\| \leq L\|x - z\|.$$

Given a monotone mapping A of C into H, (1.6) implies the following:

$$x \in \mathsf{VI}(C, A) \Rightarrow x = P_C(x - \lambda Ax),\ \forall\lambda > 0,$$

and

$$x = P_C(x - \lambda Ax),\ \text{for some}\ \lambda > 0 \Rightarrow x \in \mathsf{VI}(C, A).$$

It is well known that H satisfies the Opial's condition [**?**], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - z\|\ \ \forall z \in H\ \ \text{with}\ \ x \neq z.$$

Observe that the class of k-strictly pseudo-contractive mappings includes as a sub class of the class of nonexpansive mappings i.e., when $k = 0$. The mapping T is as well said to be pseudo-contractive if $k = 1$, and T is said to be strongly pseudo-contractive if there exists $k \in (0, 1)$ such that $T - kI$ is pseudo-contractive.

Definition 1.5. *A set valued Mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply that $\langle x - y, f - g\rangle \geqslant 0$. A monotone mapping $T : H \to 2^H$ is said to be maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping, and we say T is maximal monotone.*

It is well known that a mapping $T : H \to 2^H$ is maximal monotone if and only if for any $(x, f) \in H \times H,\ \langle x - y, f - g\rangle \geqslant 0$ for every $(y, g) \in G(T)$ imply that $f \in Tx$.
Given A a monotone mapping of C into H and $N_C w$ the normal cone to C at $w \in C$,i.e., $N_C w = \{x \in H : \langle w - y, x\rangle \geqslant 0,\ \forall y \in C\}$ and define

$$T(w) = \begin{cases} Aw + N_C w, & w \in C, \\ \emptyset, & w \in C. \end{cases}$$

Then T is maximal monotone and $0 \in Tw$ if and only if $w \in$VI(C, A); see[6].

Many studies have been done on iterative methods for nonexpansive mappings in the literature, see ([7],[8]), but that of strictly pseudo-contractive maps are far less developed because the second term appearing in the right hand side of (1.4) posses a lot of treat in computations. However, in 1967, Browder and Petryshyn initiated the study of fixed point of strictly pseudo-contractive maps in their work. Since strictly pseudo-contractive maps is one of the most important class of mappings in nonlinear mappings, and has more interesting and powerful applications in solving inverse problems see Scherzer [9], it is of high importance to develop iterative methods for strictly pseudo-contractive maps. Recently, see ([10],[11],[12],[13]), many authors have devoted time in developing schemes for finding fixed points for strictly pseudo-contractive maps.

Some methods, see ([3],[14],[15],[16],[17]), have been proposed by many authors to solve the problem (1.2). Also, some authors, see ([18],[19]), have proposed iterative methods for finding common element of fixed point set of nonexpansive mappings and the set of solutions to the variational inequality for monotone, Lipschitz continuous mappings, the set of solution to a system of equilibrium problems.

Combining the Mann iteration technique, the extragradient methods for variational inequality and system of equilibrium problems proposed by Korpelevich in [2], Jian-Wen Peng, Soon-Yi Wu, Gang-Lun Fan in [4] as well as Yonghong Yao, Yeong-Cheng Liou, Jen-Chih Yao in [20]. We consider a new iterative scheme for finding a common element of the set of solution to a system of equilibrium problems, the fixed point set of a k-strictly pseudo-contractive map and the set of solutions to the variational inequality for a monotone, Lipschitz continuous mappings. We obtain a strong convergence result for the sequence generated by our scheme. The results in this paper generalize and improve so many well known results in the literature.

2 Preliminaries

We present, in this section, some useful lemmas that will be used to prove our main results.

Lemma 2.1. *Let H be a real Hilbert space. Then the following inequality holds;*

$$\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2 \tag{2.1}$$

for all $x, y \in H$.

Lemma 2.2. *Let H be a real Hilbert space. Then the following inequality holds;*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle \tag{2.2}$$

for all $x, y \in H$.

Lemma 2.3. *[21] Let H be a real Hilbert space. Then the following inequality holds;*

$$\|\lambda x+(1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.3}$$

for all $x, y \in H$, $\lambda \in [0,1]$.

Lemma 2.4. *[22] Let $\{a_n\}$ be a sequence of non negative real numbers such that*

$$a_{n+1} \leqslant (1-\sigma_n)a_n + \sigma_n\eta_n + \delta_n, \qquad n \geqslant 1 \tag{2.4}$$

where
(i) $\{\sigma_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \sigma_n = \infty$;
(ii) $\limsup_{n\to\infty} \eta_n \leqslant 0$;
(iii) $\delta_n \geqslant 0$, $n \geqslant 1$, $\sum_{n=0}^{\infty} \delta_n < \infty$.
Then,

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 2.5. *[23] Let X be a Banach space, $\{x_n\}, \{y_n\}$ be two bounded sequences in X and $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$$

Suppose that $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \geq 1$ *and*

$$\limsup_{n\to\infty}\{\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\} \leq 0,$$

then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

In order to solve the equilibrium problem, we assume that the bifunction F satisfies the following conditions imposed in [3]:
$(A1)$ $F(x,x) = 0$ for all $x \in C$;
$(A2)$ F is monotone, $i.e$ $F(x,y) + F(y,x) \leqslant 0$ for any $x, y \in C$,
$(A3)$ For each $x, y, z \in C$,

$$\lim_{t\to 0} F(tz+(1-t)x, y) \leqslant F(x,y);$$

$(A4)$ For each $x \in C$, $y \longmapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.6. *[3]Let C be a nonempty, closed and convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. Let $x \in H$ and $r > 0$. Then there exists $c \in C$ such that*

$$F(c, y) + \frac{1}{r}\langle y - c, c - x\rangle \geqslant 0$$

for all $y \in C$.

Lemma 2.7. *[1]Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. For $x \in H$ and $r > 0$, define a mapping $T_r^F : H \longrightarrow C$ as follows:*

$$T_r^F(x) = \left\{ c \in C : F(c, y) + \frac{1}{r}\langle y - c, c - x\rangle \geqslant 0, \forall y \in C \right\}$$

for all $x \in H$. We then have that the following statements hold:
(1) *T_r^F is singled-valued;*
(2) *T_r^F is firmly nonexpansive, $i.e$, for any $x, y \in H$,*

$$\|T_r^F(x) - T_r^F(y)\|^2 \leqslant \langle T_r^F(x) - T_r^F(y), x - y\rangle;$$

(3) *Fix $(T_r^F) = EP(F)$;*
(4) *$EP(F)$ is closed and convex.*

Lemma 2.8. *[24]Let $T : C \longrightarrow H$ be $\tilde{k}$-strictly pseudo-contractive mapping.*
Define $S : C \longrightarrow H$ by

$$Sx = \alpha x + (1 - \alpha)Tx$$

for each $x \in C$.
Then, as $\alpha \in [\tilde{k}, 1)$, S is nonexpansive such that $Fix(S) = Fix(T)$. We call S the S-mapping generated by T.

Lemma 2.9 (*Demi-closed principle*)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strictly pseudo-contractive mapping. Then $I - T$ is demi-closed at 0, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

3 Main Results

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. For each $k = 1, 2, \cdots, m$, let F_k be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ and A be a strongly monotone and L-Lipschitz continuous mapping of C into H. Let $T : C \longrightarrow C$ be a $\tilde{k}$-strictly pseudo-contractive mapping and S be the S-mapping generated by T, such that $\Omega := Fix(T) \cap VI(A, C) \cap \left(\bigcap_{k=1}^m EP(F_k)\right) \neq \emptyset$. Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by $u, x_1 \in C$,*

$$\begin{cases} u_n = T_{r_m,n}^{F_m} T_{r_{m-1},n}^{F_{m-1}} \cdots T_{r_2,n}^{F_2} T_{r_1,n}^{F_1} x_n, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ q_n = P_C(u_n - \lambda_n A y_n), \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n S q_n - \alpha_n(x_n - u) \end{cases} \tag{3.1}$$

for all $n = 1, 2, \cdots$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{r_{k,n}\}$, $k \in \{1, 2, \cdots, m\}$ are sequences of real numbers satisfying the following conditions:
$(B1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

$(B2)$ $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;
$(B3)$ $(\lambda_n)_n \subset \left(0, \frac{1}{L}\right)$, $\lim_{n\to\infty} \lambda_n = 0$;
$(B4)$ $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1, 2, \cdots, m\}$.
Then the sequences $\{x_n\}$, $\{u_n\}$, $\{q_n\}$ *and* $\{y_n\}$ *converge strongly to the common point* $x^* \in \Omega$, *given by* $x^* = P_\Omega(u)$.

Proof. We shall divide the proof into 8 steps as follows:

Step 1: We show that the sequence $\{x_n\}$ is bounded.
Let $p \in F(T)$. We take $G_n^k = T_{r_{k,n}}^{F_k} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$ for each $k \in \{1, 2, \cdots, m\}$ and $G_n^0 = I$ for all n, hence $u_n = G_n^m x_n$.
Let $p \in \Omega$. By (3) of Lemma2.7 for each $k \in \{1, 2, \cdots, m\}$ $T_{r_{k,n}}^{F_k}$ is nonexpansive and p is a fixed point of $T_{r_{k,n}}^{F_k}$, we have that

$$\|u_n - p\| = \|G_n^m x_n - G_n^m p\| \le \|x_n - p\|, \quad \forall n \in \mathbb{N} \tag{3.2}$$

By using (1.7), the fact that A is monotone and that $p \in \mathrm{VI}(C, A)$, we have the following

$$\begin{aligned}
\|q_n - p\|^2 &\le \|u_n - \lambda_n A y_n - p\|^2 - \|u_n - \lambda_n A y_n - q_n\|^2 \\
&= \|u_n - p\|^2 - \|q_n - u_n\|^2 + 2\lambda_n \langle A y_n, p - q_n\rangle \\
&= \|u_n - p\|^2 - \|q_n - u_n\|^2 + 2\lambda_n(\langle A y_n - Ap, p - y_n\rangle + \langle Ap, p - y_n\rangle \\
&\quad + \langle A y_n, y_n - q_n\rangle) \\
&\le \|u_n - p\|^2 - \|q_n - u_n\|^2 + 2\lambda_n \langle A y_n, y_n - q_n\rangle \\
&\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - q_n\rangle - \|y_n - q_n\|^2 \\
&\quad + 2\lambda_n \langle A y_n, y_n - q_n\rangle \\
&= \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - q_n\|^2 + 2\langle u_n - \lambda_n A y_n - y_n, q_n - y_n\rangle.
\end{aligned}$$

Now, $y_n = P_C(u_n - \lambda_n A u_n)$ and A L-Lipschitz continuous gives that

$$\begin{aligned}
\langle u_n - \lambda_n A y_n - y_n, q_n - y_n\rangle &= \langle u_n - \lambda_n A u_n - y_n, q_n - y_n\rangle + \langle \lambda_n A u_n - \lambda_n A y_n, q_n - y_n\rangle \\
&\le \langle \lambda_n A u_n - \lambda_n A y_n, q_n - y_n\rangle \\
&\le \lambda_n L \|u_n - y_n\| \|q_n - y_n\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|q_n - p\|^2 &\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - q_n\|^2 + 2\lambda_n L \|u_n - y_n\| \|q_n - y_n\| \\
&\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - q_n\|^2 + \lambda_n^2 L^2 \|u_n - y_n\|^2 + \|q_n - y_n\|^2 \\
&= \|u_n - p\|^2 + (\lambda_n^2 L^2 - 1)\|u_n - y_n\|^2 \\
&\le \|u_n - p\|^2
\end{aligned} \tag{3.3}$$

Now using the fact that A is $L-$ Lipschitz continuous and monotone, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\leqslant \|u_n - p - \lambda_n(Ap - A u_n)\|^2 \\
&\leqslant \|u_n - p\|^2 + L^2 \lambda_n^2 \|u_n - p\|^2 + 2L\lambda_n \|u_n - p\|^2 \\
&= (1 + L\lambda_n)^2 \|u_n - p\|^2.
\end{aligned}$$

Hence by (3.2) we have

$$\|y_n - p\| \leqslant (1 + L\lambda_n)\|x_n - p\| \quad \forall n \geqslant 1. \tag{3.4}$$

We have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\beta_n)x_n+\beta_n Sq_n-\alpha_n(x_n-u)-p\| \\
&= \|(1-\beta_n)(x_n-p)+\beta_n(Sq_n-p)-\alpha_n(x_n-u)\| \\
&= \|(1-\beta_n)(x_n-p)+\beta_n(Sq_n-p)-\alpha_n(x_n-p)+\alpha_n(u-p)\| \\
&= \|(1-\beta_n-\alpha_n)(x_n-p)+\beta_n(Sq_n-p)+\alpha_n(u-p)\|.
\end{aligned}$$

Using (3.2), (3.3) and Lemma2.8 we have

$$\|p-Sq_n\|^2=\|Sp-Sq_n\|^2\leq\|p-q_n\|^2\leq\|x_n-p\|^2. \tag{3.5}$$

Hence

$$\begin{aligned}
\|x_{n+1}-p\| &\leqslant (1-\beta_n-\alpha_n)\|x_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|(u-p)\| \\
&= (1-\alpha_n)\|x_n-p\|+\alpha_n\|u-p\| \\
&\leq \max\{\|x_n-p\|,\|u-p\|\}.
\end{aligned} \tag{3.6}$$

Inductively, we get

$$\|x_n-p\|\leqslant\max\{\|x_1-p\|,\|u-p\|\}.$$

Hence, $\{x_n\}$ is bounded. From (3.2), (3.3) and (3.4) we as well obtain that $\{u_n\}$,$\{q_n\}$ and $\{y_n\}$ are bounded.

Since the mapping A is Lipschitz continuous, we also obtain the boundedness of the sequences $\{Ax_n\}$, $\{Au_n\}$, $\{Ay_n\}$.

Also, since S is nonexpansive, we obtain that $\{Sx_n\}$ and $\{Sq_n\}$ are bounded, using Lemma 2.8. Indeed,

$$\|Sx_n-p\|=\|Sx_n-Sp\|\leqslant\|x_n-p\| \tag{3.7}$$

and

$$\|Sq_n-p\|=\|Sq_n-Sp\|\leqslant\|q_n-p\|. \tag{3.8}$$

Hence, boundedness of $\{Sx_n\}$ and $\{Sq_n\}$ follows from the boundedness of $\{x_n\}$ and $\{q_n\}$ respectively.

Step 2

Let $\{s_n\}$ be a bounded sequence in C. We shall show that

$$\lim_{n\to\infty}\|G_n^m s_n-G_{n+1}^m s_n\|=0. \tag{3.9}$$

By step 2 of the proof of Theorem 3.1 in [25], it follows that for any $k\in\{1,2,\cdots,m\}$,

$$\lim_{n\to\infty}\|T_{r_k,n+1}^{F_k}s_n-T_{r_k,n}^{F_k}s_n\|=0. \tag{3.10}$$

Using condition 2 of Lemma 2.7 ($T_{r_k,n}^{F_k}$ is nonexpansive) and the definition of G_n^m, we have

$$\begin{aligned}
\|G_n^m s_n-G_{n+1}^m s_n\| &= \|T_{r_m,n}^{F_m}G_n^{m-1}s_n-T_{r_m,n+1}^{F_m}G_{n+1}^{m-1}s_n\| \\
&\leq \|T_{r_m,n}^{F_m}G_n^{m-1}s_n-T_{r_m,n+1}^{F_m}G_n^{m-1}s_n\|+\|T_{r_m,n+1}^{F_m}G_n^{m-1}s_n-T_{r_m,n+1}^{F_m}G_{n+1}^{m-1}s_n\| \\
&\leq \|T_{r_m,n}^{F_m}G_n^{m-1}s_n-T_{r_m,n+1}^{F_m}G_n^{m-1}s_n\|+\|G_n^{m-1}s_n-G_{n+1}^{m-1}s_n\| \\
&\leq \|T_{r_m,n}^{F_m}G_n^{m-1}s_n-T_{r_m,n+1}^{F_m}G_n^{m-1}s_n\|+\|T_{r_{m-1},n}^{F_{m-1}}G_n^{m-2}s_n-T_{r_{m-1},n+1}^{F_{m-1}}G_n^{m-2}s_n\| \\
&\quad +\|G_n^{m-2}s_n-G_{n+1}^{m-2}s_n\| \\
&\leq \|T_{r_m,n}^{F_m}G_n^{m-1}s_n-T_{r_m,n+1}^{F_m}G_n^{m-1}s_n\|+\|T_{r_{m-1},n}^{F_{m-1}}G_n^{m-2}s_n-T_{r_{m-1},n+1}^{F_{m-1}}G_n^{m-2}s_n\| \\
&\quad +\cdots+\|T_{r_2,n}^{F_2}G_n^1 s_n-T_{r_2,n+1}^{F_2}G_n^1 s_n\|+\|T_{r_1,n}^{F_1}s_n-T_{r_1,n}^{F_1}s_n\|.
\end{aligned}$$

from which (3.9) follows by (3.10).

Step 3: $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$
We know that $u_n = G_n^m x_n$ and $u_{n+1} = G_{n+1}^m x_{n+1}$. We then have that

$$\begin{aligned}\|u_n - u_{n+1}\| &= \|G_n^m x_n - G_{n+1}^m x_{n+1}\| \\ &\leq \|G_n^m x_n - G_{n+1}^m x_n\| + \|G_{n+1}^m x_n - G_{n+1}^m x_{n+1}\| \\ &\leq \|G_n^m x_n - G_{n+1}^m x_{n+1}\| + \|x_n - x_{n+1}\|. \end{aligned} \tag{3.11}$$

Now observe that

$$\begin{aligned}\|q_{n+1}-q_n\| &= \|P_C(u_{n+1}-\lambda_{n+1}Ay_{n+1}) - P_C(u_n - \lambda_n Ay_n)\| \\ &\leq \|(u_{n+1}-\lambda_{n+1}Ay_{n+1}) - (u_n - \lambda_n Ay_n)\| \\ &= \|u_{n+1}-u_n - \lambda_{n+1}(Au_{n+1}-Au_n) + \lambda_{n+1}(Au_{n+1}-Ay_{n+1}-Au_n) + \lambda_n Ay_n)\| \\ &\leq \|u_{n+1}-u_n\| + \lambda_{n+1}\|Au_{n+1}-Au_n\| + \lambda_{n+1}\|Au_{n+1}-Ay_{n+1}-Au_n\| + \lambda_n\|Ay_n\| \\ &\leq \|u_{n+1}-u_n\| + L\lambda_{n+1}\|Au_{n+1}-Au_n\| + \lambda_{n+1}\|Au_{n+1}-Ay_{n+1}-Au_n\| + \lambda_n\|Ay_n\| \\ &\leq \|u_{n+1}-u_n\| + (\lambda_{n+1}+\lambda_n)M, \end{aligned} \tag{3.12}$$

where M is a constant such that

$$M \geqslant \sup_{n\geqslant 1}\{k\|Au_{n+1}-Au_n\| + \|Au_{n+1}-Ay_{n+1}-Au_n\| + \|Ay_n\|\}.$$

Hence, from (3.11) and (3.12) we have

$$\|q_{n+1}-q_n\| \leq \|G_n^m x_n - G_{n+1}^m x_n\| + \|x_n - x_{n+1}\| + (\lambda_{n+1}+\lambda_n)M.$$

We define the sequence $\{z_n\}$ to be such that for any $n \geqslant 1$,

$$\beta_n z_n = x_{n+1} - (1-\beta_n)x_n. \tag{3.13}$$

Therefore

$$\begin{aligned}\|z_{n+1}-z_n\| &= \left\|\frac{x_{n+2}-(1-\beta_{n+1})x_{n+1}}{\beta_{n+1}} - \frac{x_{n+1}-(1-\beta_n)x_n}{\beta_n}\right\| \\ &= \|\frac{(1-\beta_{n+1})x_{n+1}+\beta_{n+1}Sq_{n+1}-\alpha_{n+1}(x_{n+1}-u)-(1-\beta_{n+1})x_{n+1}}{\beta_{n+1}} \\ &\quad -\frac{(1-\beta_n)x_n+\beta_n Sq_n - \alpha_n(x_n-u)-(1-\beta_n)x_n}{\beta_n}\| \\ &= \left\|Sq_{n+1}-Sq_n - \frac{\alpha_{n+1}}{\beta_{n+1}}(x_{n+1}-u) + \frac{\alpha_n}{\beta_n}(x_n-u)\right\| \\ &\leqslant \|q_{n+1}-q_n\| + \frac{\alpha_{n+1}}{\beta_{n+1}}\|u-x_{n+1}\| + \frac{\alpha_n}{\beta_n}\|u-x_n\|. \end{aligned} \tag{3.14}$$

Hence, by (3.13) and (3.15), we obtain

$$\begin{aligned}\|z_{n+1}-z_n\| - \|x_n - x_{n+1}\| &\leqslant \|G_n^m x_n - G_{n+1}^m x_n\| + \frac{\alpha_{n+1}}{\beta_{n+1}}\|u-x_{n+1}\| + \frac{\alpha_n}{\beta_n}\|u-x_n\| \\ &\quad +(\lambda_{n+1}+\lambda_n)M. \end{aligned}$$

Therefore, by conditions $(B1)$-$(B3)$ we have

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_n - x_{n+1}\|) \leqslant 0.$$

By Lemma 2.5, we then have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0,$$

from which it follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\beta_n\|z_n - x_n\| = 0. \tag{3.15}$$

We obtain also from (3.9), (3.11), (3.12) and (3.15) that

$$\lim_{n\to\infty}\|q_{n+1} - q_n\| = 0$$

and

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0.$$

Step 4: $\lim_{n\to\infty}\|u_n - q_n\|= 0.$
We know that

$$x_{n+1} = (1-\beta_n)x_n + \beta_n Sq_n - \alpha_n(x_n - u). \tag{3.16}$$

Observe that

$$\begin{aligned}\|x_n - Sq_n\| &\leqslant \|x_{n+1} - x_n\| + \|x_{n+1} - Sq_n\|\\ &= \|x_{n+1} - x_n\| + \|(1-\beta_n)x_n + \beta_n Sq_n - \alpha_n(x_n - u) - Sq_n\|\\ &\leqslant \|x_{n+1} - x_n\| + (1-\beta_n)\|x_n - Sq_n\| + \alpha_n\|x_n - u\|. \end{aligned}\tag{3.17}$$

Hence,

$$\beta_n\|x_n - Sq_n\| \leqslant \|x_{n+1} - x_n\| + \alpha_n\|x_n - u\|. \tag{3.18}$$

It follows from conditions $(B1)$ and $(B2)$ that

$$\lim_{n\to\infty}\|x_n - Sq_n\|= 0. \tag{3.19}$$

Now

$$\begin{aligned}\|y_n - q_n\| &\leqslant \|P_C(u_n - \lambda_n Ay_n) - P_C(u_n - \lambda_n Au_n)\|\\ &\leqslant \|u_n - \lambda_n Ay_n - u_n - \lambda_n Au_n\|\\ &\leqslant \lambda_n L\|y_n - u_n\|\\ &\leqslant \lambda_n LM_1, \quad \text{for some } M_1 > 0. \end{aligned}\tag{3.20}$$

Therefore by condition $(B3)$, we have

$$\lim_{n\to\infty}\|y_n - q_n\|= 0. \tag{3.21}$$

For all $p \in \Omega$, by using Lemma (2.8), (3.2) and (3.3), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leqslant \|(1-\beta_n)x_n + \beta_n Sq_n - \alpha_n(x_n - u) - p\|^2\\ &\leqslant \|(1-\beta_n)x_n + \beta_n Sq_n - p\|^2 + 2\alpha_n\|x_n - u\|\|(1-\beta_n)x_n + \beta_n Sq_n - p\|\\ &\quad +\alpha_n^2\|x_n - u\|^2\\ &\leqslant (1-\beta_n)\|x_n - p\|^2 + \beta_n\|Sq_n - p\|^2\\ &\quad +2\alpha_n\|x_n - u\|\|(1-\beta_n)x_n + \beta_n Sq_n - p\| + \alpha_n^2\|x_n - u\|^2\\ &\leqslant (1-\beta_n)\|x_n - p\|^2 + \beta_n\|q_n - p\|^2 + 2\alpha_n\|x_n - u\|\|(1-\beta_n)x_n\\ &\quad +\beta_n Sq_n - p\| + \alpha_n^2\|x_n - u\|^2\\ &\leqslant (1-\beta_n)\|x_n - p\|^2 + \beta_n\left[\|x_n - p\|^2 - (1-\lambda_n^2L^2)\|u_n - y_n\|^2\right]\\ &\quad +2\alpha_n\|x_n - u\|\|(1-\beta_n)x_n + \beta_n Sq_n - p\| + \alpha_n^2\|x_n - u\|^2. \end{aligned}\tag{3.22}$$

It follows that

$$\begin{aligned}\|u_n-y_n\|^2 \quad &\leqslant \quad \frac{1}{1-\lambda_n^2L^2}[(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\| \\ &\quad +\alpha_n^2\|x_n-u\|\|(1-\beta_n)x_n+\beta Sq_n-p\|+\alpha_n^2\|x_n-u\|^2]. \end{aligned} \tag{3.23}$$

By condition $(B1)$ and step 3, we have

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.24}$$

Hence, since

$$\|u_n-q_n\|\leqslant\|u_n-y_n\|+\|y_n-q_n\|, \tag{3.25}$$

we have

$$\lim_{n\to\infty}\|u_n-q_n\|=0. \tag{3.26}$$

Step 5: $\lim_{n\to\infty}\|G_n^kx_n-G_n^{k-1}x_n\|=0$, $k=1,2,\cdots,m$.
Let $p\in\Omega$. Firmly nonexpansiveness of $T_{r_k,n}^{F_K}$ for each $k=1,2,\cdots,m$ gives

$$\begin{aligned}\|G_n^kx_n-p\|^2 &= \|T_{r_k,n}^{F_k}G_n^{k-1}x_n-T_{r_k,n}^{F_k}p\|^2 \\ &\leqslant \langle G_n^kx_n-p,G_n^{k-1}x_n-p\rangle \\ &= \frac{1}{2}\left(\|G_n^kx_n-p\|^2+\|G_n^{k-1}x_n-p\|^2-\|G_n^kx_n-G_n^{k-1}x_n\|^2\right).\end{aligned} \tag{3.27}$$

Therefore, we obtain that

$$\|G_n^kx_n-p\|^2 \quad \leqslant \quad \|G_n^{k-1}x_n-p\|^2-\|G_n^kx_n-G_n^{k-1}x_n\|^2, \quad for \quad k=1,2,\cdots,m, \tag{3.28}$$

which implies that for each $k\in\{1,2,\cdots,m\}$,

$$\begin{aligned}\|G_n^kx_n-p\|^2 &\leqslant \|G_n^0x_n-p\|^2-\|G_n^kx_n-G_n^{k-1}x_n\|^2-\|G_n^{k-1}x_n-G_n^{k-2}x_n\|^2 \\ &\quad -\cdots-\|G_n^2x_n-G_n^1x_n\|^2-\|G_n^1x_n-G_n^0x_n\|^2 \\ &\leqslant \|x_n-p\|^2-\|G_n^kx_n-G_n^{k-1}x_n\|^2.\end{aligned} \tag{3.29}$$

Now, using the fact that $u_n=G_n^mx_n$, Lemma 2.2 and (3.3) , we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\beta_n)x_n+\beta_nSq_n-\alpha_n(x_n-u)-p\|^2 \\ &\leqslant \|(1-\beta_n-\alpha_n)(x_n-p)+\beta_n(Sq_n-p)\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\ &= \|[1-(\beta_n+\alpha_n)](x_n-p)+(\alpha_n+\beta_n)(Sq_n-p)-\alpha_n(Sq_n-p)\|^2 \\ &\quad +2\alpha_n\langle u-p,x_{n+1}-p\rangle \\ &\leqslant [1-(\beta_n+\alpha_n)]\|x_n-p\|^2+(\alpha_n+\beta_n)\|u_n-p\|^2 \\ &\quad +2\alpha_n\langle p-Sq_n,x_{n+1}-p\rangle+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\ &\leqslant [1-(\beta_n+\alpha_n)]\|x_n-p\|^2+(\alpha_n+\beta_n)\|G_n^kx_n-p\|^2 \\ &\quad +2\alpha_n\langle p-Sq_n,x_{n+1}-p\rangle+2\alpha_n\langle u-p,x_{n+1}-p\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}(\alpha_n+\beta_n)\|G_n^kx_n-G_n^{k-1}x_n\|^2 &\leqslant \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\ &\quad +2\alpha_n[\|x_{n+1}-p\|\|x_n-p\|+\|u-p\|\|x_{n+1}-p\|] \\ &\leqslant (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\| \\ &\quad +2\alpha_n[\|x_{n+1}-p\|\|x_n-p\|+\|u-p\|\|x_{n+1}-p\|].\end{aligned}$$

Using condition $(B1)$ and that $\|x_{n+1}-x_n\|\to 0$, we obtain that

$$\lim_{n\to\infty}\|G_n^kx_n-G_n^{k-1}x_n\|=0. \tag{3.30}$$

Step 6: $\lim_{n\to\infty}\|Sy_n - y_n\|= 0.$
Observe that

$$\begin{aligned}\|Sy_n - y_n\| &\leqslant \|Sy_n - Sq_n\|+\|Sq_n - x_n\|+\|G_n^0x_n - G_n^1x_n\|+\|G_n^1x_n - G_n^2x_n\| \\ &\quad + \cdots + \|G_n^{m-1}x_n - G_n^m x_n\|+\|u_n - y_n\| \\ &\leqslant \|y_n - q_n\|+\|Sq_n - x_n\|+\|G_n^0x_n - G_n^1x_n\|+\|G_n^1x_n - G_n^2x_n\| \\ &\quad + \cdots + \|G_n^{m-1}x_n - G_n^m x_n\|+\|u_n - y_n\|. \end{aligned} \tag{3.31}$$

Therefore,

$$\lim_{n\to\infty}\|Sy_n - y_n\|= 0. \tag{3.32}$$

Step 7: $\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle \leqslant 0$ where $x^* = P_\Omega(u)$,
i.e.,$\langle x^* - u, z - x^*\rangle \geqslant 0 \;\; \forall z \in \Omega.$
Let $\{x_{n_j}\}$ be a subsequense of $\{x_n\}$ such that

$$\lim_{j\to\infty}\langle u - x^*, x_{n_j} - x^*\rangle = \limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle. \tag{3.33}$$

Using the boundedness of $\{x_n\}$, there exist a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_k}} \rightharpoonup w$.
Without loss of generality, we assume that $x_{n_j} \rightharpoonup w.$
We show that $w \in \Omega$.
By step 5,

$$\|G_n^k x_n - G_n^{k-1}x_n\| \to 0 \quad for \quad each \quad k = 1, 2, \cdots, m,$$

which implies that $G_n^k x_{n_j} \rightharpoonup w$ for each $k = 1, 2, \cdots, m.$
Since $\|u_n - y_n\| \to 0$ and $\|u_n - q_n\| \to 0$, we have that $u_{n_j} \rightharpoonup w$, $y_{n_j} \rightharpoonup w$ and $q_{n_j} \rightharpoonup w.$
Also, $\{u_{n_j}\} \subset C$ and C is closed and convex implies that $w \in C$.
(i) We show that $w \in Fix(S) = Fix(T)$.
By Lemma 2.9 and step 6, we obtain that $w \in F(S) = F(T)$.
(ii) We show that $w \in \bigcap_{k=1}^m EP(F_k)$.
By Lemma 2.6, for each $k = 1, 2, \cdots, m$, we have

$$F_k\left(G_n^k x_n, y\right) + \frac{1}{r_n}\left\langle y - G_n^k x_n, G_n^k x_n - G_n^{k-1}x_n\right\rangle \geqslant 0, \; \forall y \in C.$$

It follows from $(A2)$ that

$$\frac{1}{r_n}\left\langle y - G_n^k x_n, G_n^k x_n - G_n^{k-1}x_n\right\rangle \geqslant F_k\left(y, G_n^k x_n\right), \; \forall y \in C.$$

Therefore, for each $k = 1, 2, \cdots, m$, we have

$$\left\langle y - G_{n_j}^k x_{n_j}, \frac{G_{n_j}^k x_{n_j} - G_{n_j}^{k-1}x_{n_j}}{r_{n_j}}\right\rangle \geqslant F_k\left(y, G_{n_j}^k x_{n_j}\right), \; \forall y \in C.$$

By $(A4)$, we have

$$\frac{G_{n_j}^k x_{n_j} - G_{n_j}^{k-1}x_{n_j}}{r_{n_j}} \longrightarrow 0 \quad and \quad G_{n_j}^k x_{n_j} \rightharpoonup w.$$

Hence, for each $k = 1, 2, \cdots, m$,

$$F_k(y, w) \leqslant 0, \; \forall y \in C.$$

Define

$$y_s = (1 - s)w + sy, \qquad \forall y \in C \quad and \quad s \in (0, 1].$$

By the convexity of C, we have that $y_s \in C$ from which it follows that

$$F_k(y_s, w) \leqslant 0, \text{ for each } k = 1, 2, \cdots, m.$$

Using $(A4)$, we have

$$\begin{aligned} 0 = F_k(y_s, y_s) &\leqslant sF_k(y_s, y) + (1-s)F_k(y_s, w) \\ &\leqslant sF_k(y_s, y). \end{aligned}$$

Since $s \neq 0$, we obtain that

$$F_k(y_s, y) \geqslant 0.$$

By (A3), for any $y \in C$, $F_k(w, y) = \lim_{s\to\infty} F_k(y_s, y) \geqslant 0$, for each $k = 1, 2, \cdots, m$.
Therefore for each $k = 1, 2, \cdots, m$, $w \in$EP(F_k), which gives that

$$w \in \bigcap_{k=1}^{m} \text{EP}(F_k).$$

(iii) We show that $w \in$VI(C, A).
Define the set-valued mapping T from H to 2^H i.e., $T : H \to 2^H$ by

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1, & w_1 \in C, \\ \emptyset, & w_1 \notin C. \end{cases}$$

Where N_C is the normal cone to C at $w_1 \in C$.
The mapping T in this case is maximal monotone, and $0 \in Tw_1$ if and only if $w_1 \in$VI(C, A), see[6].
Let $(w_1, h) \in G(T)$. It follows that $Tw_1 = Aw_1 + N_C w_1$, therefore $h - Aw_1 \in N_C w_1$. Hence we obtain that $\langle w_1 - s, h - Aw_1\rangle \geqslant 0$ for any $s \in C$. Since $q_n = P_C(u_n - \lambda_n Ay_n)$ and $w_1 \in C$, we have

$$\langle u_n - \lambda_n Ay_n - q_n, q_n - w_1\rangle \geqslant 0.$$

Therefore

$$\langle w_1 - q_n, \frac{q_n - u_n}{\lambda_n} + Ay_n\rangle \geqslant 0 \ \text{ for each } n \geqslant 1.$$

Hence

$$\begin{aligned} \langle q_{n_j} - w_1, h\rangle &\leqslant \langle q_{n_j} - w_1, Aw_1\rangle \\ &\leqslant \langle q_{n_j} - w_1, Aw_1\rangle - \langle q_{n_j} - w_1, \frac{q_{n_j} - u_{n_j}}{\lambda_{n_j}} + Ay_{n_j}\rangle \\ &= \langle q_{n_j} - w_1, Aw_1 - Ay_{n_j} - \frac{q_{n_j} - u_{n_j}}{\lambda_{n_j}}\rangle \\ &= \langle q_{n_j} - w_1, Aw_1 - Aq_{n_j}\rangle + \langle q_{n_j} - w_1, Aq_{n_j} - Ay_{n_j}\rangle \\ &\quad -\langle q_{n_j} - w_1, \frac{q_{n_j} - u_{n_j}}{\lambda_{n_j}}\rangle. \end{aligned}$$

So

$$\langle q_{n_j} - w_1, h\rangle \leqslant \langle q_{n_j} - w_1, Aq_{n_j} - Ay_{n_j}\rangle - \langle q_{n_j} - w_1, \frac{q_{n_j} - u_{n_j}}{\lambda_{n_j}}\rangle.$$

Hence it follows that

$$\langle w_1 - w, h\rangle \geqslant 0.$$

Since T is maximal monotone, we have $w \in T^{-1}0$, and it follows that $w \in$VI(C, A).
(i),(ii) and (iii) give that $w \in \Omega$.
Therefore

$$\begin{aligned} \limsup_{n\to\infty} \langle u - x^*, x_n - x^*\rangle &= \lim_{j\to\infty} \langle u - x^*, x_{n_j} - x^*\rangle \\ &= \langle u - x^*, w - x^*\rangle \leqslant 0. \end{aligned} \tag{3.34}$$

Step 8: $\lim_{n\to\infty}\|x_n - x^*\|= 0.$

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n)x_n+\beta_n Sq_n-\alpha_n(x_n-u)-x^*\|^2\\
&= \|(1-\beta_n)(x_n-x^*)+\beta_n(Sq_n-x^*)-\alpha_n(x_n-x^*)+\alpha_n(u-x^*)\|^2\\
&= \|(1-\beta_n-\alpha_n)(x_n-x^*)+\beta_n(Sq_n-x^*)+\alpha_n(u-x^*)\|^2\\
&\leqslant \|(1-\beta_n-\alpha_n)(x_n-x^*)+\beta_n(Sq_n-x^*)\|^2+2\alpha_n\langle u-x^*,x_{n+1}-x^*\rangle\\
&\leqslant (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n\langle u-x^*,x_{n+1}-x^*\rangle\\
&= (1-2\alpha_n)\|x_n-x^*\|^2+2\alpha_n\left[\frac{\alpha_n}{2}\|x_n-x^*\|^2+\langle u-x^*,x_{n+1}-x^*\rangle\right].
\end{aligned}$$

Using Lemma2.4 and (3.34), it follows that $\lim_{n\to\infty}\|x_n - x^*\| = 0.$
Using (3.3),(3.2) and (3.4) we obtain that the sequences $\{q_n\},\{u_n\}$ and $\{y_n\}$ converge to x^*. This completes the proof. ☐

4 Applications

The following are direct applications of our main result:

Corollary 4.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. For each $k = 1, 2, \cdots, m$, let F_k be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ and A be a strongly monotone and L-Lipschitz continuous mapping of C into H. Let $T : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := Fix(T)\cap VI(A,C)\cap\left(\bigcap_{k=1}^m EP(F_k)\right)\neq\emptyset$. Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by $u, x_1 \in C$,*

$$\begin{cases}
u_n = T^{F_m}_{r_m,n}T^{F_{m-1}}_{r_{m-1},n}\cdots T^{F_2}_{r_2,n}T^{F_1}_{r_1,n}x_n,\\
y_n = P_C(u_n-\lambda_n Au_n),\\
q_n = P_C(u_n-\lambda_n Ay_n),\\
x_{n+1} = (1-\beta_n)x_n+\beta_n Tq_n-\alpha_n(x_n-u)
\end{cases}\tag{4.1}$$

for all $n = 1, 2, \cdots$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{r_{k,n}\}$, $k\in\{1,2,\cdots,m\}$ are sequences of real numbers satisfying the following conditions:
$(B1)$ $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$*;*
$(B2)$ $0 < \liminf_{n\to\infty}\beta_n \leqslant \limsup_{n\to\infty}\beta_n < 1$*;*
$(B3)$ $\lim_{n\to\infty}\lambda_n = 0$*;*
$(B4)$ $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty}|r_{k,n+1}-r_{k,n}| = 0$ *for each* $k\in\{1,2,\cdots,m\}$.
Then the sequences $\{x_n\}$, $\{u_n\}$, $\{q_n\}$ and $\{y_n\}$ converge strongly to the common point $x^*\in\Omega$, given by $x^* = P_\Omega(u)$.

Proof. The conclusion follows immediately by Theorem 3.1, since we have that T is a nonexpansive mapping. ☐

Corollary 4.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be a strongly monotone and L-Lipschitz continuous mapping of C into H and $T : C \longrightarrow C$ be a nonexpansive mapping such that $\Gamma := Fix(T)\cap VI(A,C)\neq\emptyset$. Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by $u, x_1\in C$,*

$$\begin{cases}
y_n = P_C(x_n-\lambda_n Ax_n),\\
x_{n+1} = (1-\beta_n)x_n+\beta_n TP_C(x_n-\lambda_n Ay_n)-\alpha_n(x_n-u)
\end{cases}\tag{4.2}$$

for all $n = 1, 2, \cdots$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences of real numbers satisfying conditions B1-B3.
Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the common point
$x^* \in \Gamma$, given by $x^* = P_\Gamma(u)$.

Proof. Let $F_k(x, y) = 0$ for any $k \in \{1, 2, \cdots, m\}$, for all $x, y \in C$ and $r_n = 1\ \forall n \in \mathbb{N}$ in Theorem 3.1. It follows that $T^{F_m}_{r_m,n} = I$ (the identity mapping) $\forall n, m \in \mathbb{N}$, i.e., $u_n = x_n\ \forall n \in \mathbb{N}$. Clearly the conditions of Theorem 3.1 hold. Therefore we obtain that the sequence $\{x_n\}$ generated by 4.2 converges to the point $x^* \in \Gamma$ with $x^* = P_\Gamma(u)$. □

Corollary 4.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be a strongly monotone and L-Lipschitz continuous mapping of C into H. Suppose $\{x_n\}_{n=1}^{\infty}$ is iteratively generated by $u, x_1 \in C$,*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda_n A y_n) - \alpha_n(x_n - u) \end{cases} \tag{4.3}$$

for all $n = 1, 2, \cdots$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences of real numbers satisfying conditions B1-B3.
Then the sequence $\{x_n\}$ converges strongly to $x^* \in C$, given by $x^* = P_{VI(A,C)}(u)$.

Proof. Let $F_k(x, y) = 0$ for any $k \in \{1, 2, \cdots, m\}$, for all $x, y \in C$ and $r_n = 1\ \forall n \in \mathbb{N}$ in Theorem 3.1. It follows that $T^{F_m}_{r_m,n} = I$ (the identity mapping) $\forall n, m \in \mathbb{N}$, i.e., $u_n = x_n\ \forall n \in \mathbb{N}$. Letting $T = I$ (the identity mapping), clearly the conditions of Theorem 3.1 hold. Therefore we obtain that the sequence $\{x_n\}$ generated by 4.3 converges to the point $x^* \in VI(A, C)$ with $x^* = P_{VI(A,C)}(u)$. □

Remark 1. *Using Corollary 4.3, we have an iterative scheme to obtain the solution of a variational inequality problem involving a monotone and L-Lipschitz continuous mapping A.*

We apply Corollary 4.3 directly to the following variational inequality problem.

Proposition 4.1. *Let M be a $n \times n$ positive definite matrix, let C be a M-invariant closed subspace of $\mathbb{R}^n$ and $b \in C$. We define $F : C \to C$ by $F(x) = Mx + b$. Suppose $\{x_n\}_{n=1}^{\infty}$ is iteratively generated by $u, x_1 \in C$,*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda_n A y_n) - \alpha_n(x_n - u) \end{cases} \tag{4.4}$$

for all $n = 1, 2, \cdots$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences of real numbers satisfying conditions B1-B3.
Then the sequence $\{x_n\}$ converges strongly to a unique point $x^* \in C$, given by $x^* = P_{VI(A,C)}(u)$.

Proof. Since M is positive definite, we obtain that F is strongly monotone. It is also easy to see that F is a Lipschitzian mapping with Lipschitz constant $\|M\|$. The strong monotonicity of F guarantees a unique solution $x^* \in VI(F, C)$. Therefore, by Corollary 4.3, the sequence $\{x_n\}$ generated by 4.4 converges to the point $x^* \in VI(A, C)$ with $x^* = P_{VI(A,C)}(u)$. □

5 Conclusion

In this work, we have shown that the proposed scheme (3.1) converges strongly to a common point of the set of solution of equilibrium problems, variational inequality problem and the fixed point set of a k-strictly pseudo-contractive mapping in Hilbert spaces. We also gave some applications of our result. More interestingly we applied our result in solving a classical variational inequality problem.

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Competing Interests

Authors have declared that no competing interests exist.

Authors' Contributions

All authors contributed equally and significantly. All authors read and approved the final manuscript.

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