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# A New Algebraic Version of Monteiro's Four-Valued Propositional Calculus

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## Abstract

**In the XII Latin American Symposium on Mathematical Logic we presented a work introducing a Hilbert-style propositional calculus called four-valued Monteiro propositional calculus. This calculus, denoted by $\mathcal{M}_4$, is introduced in terms of the binary connectives $\Rightarrow$ (implication), $\rightarrow$ (weak implication), $\wedge$ (conjunction) and the unary ones $\sim$ (negation) and $\nabla$ (modal operator). In this paper, it is proved that $\mathcal{M}_4$ belongs to the class of standard systems of implicative extensional propositional calculi as defined by Rasiowa (1974). Furthermore, we show that the definitions of four-valued modal algebra and $\mathcal{M}_4$-algebra are equivalent and, in addition, obtain the completeness theorem for $\mathcal{M}_4$. We also introduce the notion of modal distributive lattices with implication and show that these algebras are more convenient than four-valued modal algebras for the study of four-valued Monteiro propositional calculus from an algebraic point of view. This follows from the fact that the implication $\rightarrow$ is one of its basic binary operations.**

## Keywords

**Mathematical Logic, Hilbert-Style Propositional Calculus, Non-Classical Logics, Four-Valued Monteiro Propositional Calculus**

## 1. Preliminaries

In 1978, A. Monteiro introduced four-valued modal algebras as a generalization of 3-valued Łukasiewicz algebras. These algebras raise a genuine interest both from the points of view of algebra and logic, and especially from that of Algebraic Logic. It is worth mentioning that Monteiro expressed his view that in near future these algebras would give rise to a four-valued modal logic with significant applications in computer science. His in-

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sight was essentially right, although we don't know if such applications have yet been developed. On the other hand, it is well known that Font and Rius (1990, 2000) have done an exhaustive research into four-valued modal logics. In particular, in (Font & Rius, 1990) they studied the class of abstract logics projectively generated by the class of logics defined on four-valued modal algebras by the family of their filters and they obtained an axiomatization of them by means of a Gentzen calculus. Furthermore, in (Font & Rius, 2000) they defined two sentential logics. One of them is algebrizable in the sense of (Blok & Pigozzi, 1989) and the corresponding algebras are four-valued modal algebras. The other is not algebrizable in the above sense, but its algebraic counterpart is also the class of four-valued modal algebras. These results gave a positive answer to Monteiro's conjecture. Besides, in (Bianco, 2004, 2008) we gave a Hilbert-style propositional calculus, called four-valued Monteiro propositional calculus which we would describe below. An algebraic study of four-valued modal algebras can be found in (Figallo et al., 1991, 1992, 1994, 1995) and (Loureiro, 1980, 1983a, 1983b, 1983c, 1984, 1985). Recall that:

A four-valued modal algebra is an algebra $\langle A,\wedge,\vee,\sim,\nabla,1\rangle$ of type $(2,2,1,1,0)$ such that the reduct $\langle A,\vee,\wedge,1\rangle$ is a distributive lattice with greatest element 1 satisfying these conditions:

(L1) $\sim\sim x = x$,

(L2) $\sim(x\wedge y) = \sim x \vee \sim y$,

(L3) $\sim(x\wedge \sim\nabla x) = 1$,

(L4) $x\wedge \sim x = \nabla x \wedge \sim x$.

From the definition, it follows that $A$ is a De Morgan algebra (Monteiro, 1960; Birkhoff, 1967). Besides, Loureiro (1982, 1983a) proved que that the variety $\mathcal{TM}$ of these algebras is generated by the algebra $\mathbb{M}_4 = \langle T_4,\vee,\wedge,\nabla,\sim,1\rangle$, where

1) $T_4 = \{0,a,b,1\}$ is the De Morgan algebra such that $\sim a = a$ and $\sim b = b$, being $a,b$ not comparable;

2) The operation $\nabla$ is defined by $\nabla 0 = 0$ and $\nabla x = 1$ for all $x \neq 0$.

In what follows, we will denote by $\mathfrak{TM}$ the category of four-valued modal algebras and their corresponding homomorphisms.

For the notions of universal algebra including distributive lattices, De Morgan algebras and category theory used in this paper we refer the reader to (Kalman, 1958; Birkhoff, 1967; MacLane, 1971; Burris & Sankappanavar, 1981).

The structure of the paper is as follows: In Section 1, we develop a Hilbert-style propositional calculus which we call four-valued Monteiro propositional calculus and denote by $\mathcal{M}_4$. Besides, we prove that $\mathcal{M}_4$ belongs to the class of standard systems of implicative extensional propositional calculi (Rasiowa, 1974). Furthermore, we show that the notions of four-valued modal algebra and $\mathcal{M}_4$-algebra are equivalent. Finally, we demonstrate that $\mathcal{M}_4$ is consistent and that the completeness theorem holds. From the equivalence established above, we conclude that from any algebra $\langle L,\wedge,\vee\sim,\nabla,1\rangle \in \mathcal{TM}$ we have that $\langle L,\to,\Rightarrow\wedge,\sim,\nabla,1\rangle$ is an $\mathcal{M}_4$-algebra where $x\to y = \nabla\sim x\vee y$ and $x\Rightarrow y = (x\to y)\wedge(\nabla x\to\nabla(x\wedge y))$ and conversely, from any $\mathcal{M}_4$-algebra we obtain a four-valued modal algebra by defining $x\vee y = \sim(\sim x\wedge\sim y)$. These statements and the fact that $\Rightarrow$ can be defined from $\to$ allow us to assert that $\mathcal{TM}$ and the variety $\mathcal{MDL}_i$ generated by $\mathbb{L}_4 = \langle\{0,a,b,1\},\to,\wedge,\sim,\nabla,1\rangle$ are polynomially equivalent. In Section 2, we introduce the variety $\mathcal{MDL}_i$ of modal distributive lattices with implication and we show that the category $\mathfrak{MDL}_i$ of these algebras and their corresponding homomorphisms is equivalent to $\mathfrak{TM}$. Thus, the category theory notably simplify the proof that should be performed to obtain this result by direct calculations. Hence, we determine a new equational description of four-valued modal algebras which is more convenient than the latter to study $\mathcal{M}_4$ from an algebraic point of view, since the implication $\to$ is one of its basic binary operations.

## 2. Four-Valued Monteiro Propositional Calculus

The main aim of this section is to describe the propositional calculus $\mathcal{M}_4$ and show that it has four-valued modal algebras as the algebraic counterpart. The terminology and symbols used here coincide in general with those used in (Rasiowa, 1974).

Let $L = (A^0, F)$ be a formalized language of zero order such that in the alphabet $A^0 = (V, L_0, L_1, L_2, U)$ the set

1) $V$ of propositional variables which they will be denoted by $\alpha, \beta, \gamma, \cdots$, is enumerable;
2) $L_0$ is empty;
3) $L_1$ contains two elements denoted by $\sim$ and $\nabla$ called negation sign and modal operator sign, respectively;
4) $L_2$ contains three elements denoted by $\wedge$, $\rightarrow$ and $\Rightarrow$ called conjunction sign, weak implication sign and implication sign, respectively;
5) $U$ contains two elements denoted by $(,)$.

Let $F$ be the set of all formulas over $A^0$. For any $\alpha, \beta$ in $F$, we shall write for short $\alpha \leftrightarrow \beta$ instead of $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

We assume that the set $A_l$ of logical axioms consists of all formulas of the following form, where $\alpha, \beta, \gamma$ are any formulas in $F$:

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$,

(A2) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$,

(A3) $(\alpha \wedge \beta) \rightarrow \alpha$,

(A4) $(\alpha \wedge \beta) \rightarrow \beta$,

(A5) $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$,

(A6) $\nabla \alpha \rightarrow \nabla (\alpha \wedge \alpha)$,

(A7) $\nabla (\alpha \wedge \beta) \rightarrow \nabla \beta$,

(A8) $(\alpha \rightarrow \beta) \rightarrow (\nabla (\alpha \wedge \beta) \rightarrow (\nabla (\gamma \wedge \alpha) \rightarrow \nabla ((\alpha \wedge \gamma) \wedge (\beta \wedge \gamma))))$,

(A9) $\nabla \nabla \alpha \rightarrow \nabla \alpha$,

(A10) $\alpha \rightarrow \nabla \alpha$,

(A11) $\nabla (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \nabla \beta)$,

(A12) $(\alpha \rightarrow \nabla \beta) \rightarrow \nabla (\alpha \rightarrow \beta)$,

(A13) $\nabla (\alpha \rightarrow (\beta \wedge \gamma)) \rightarrow \nabla ((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma))$,

(A14) $(\beta \rightarrow \gamma) \rightarrow ((\nabla \alpha \rightarrow \nabla (\alpha \wedge \beta)) \rightarrow ((\nabla \beta \rightarrow \nabla (\beta \wedge \gamma)) \rightarrow (\nabla \alpha \rightarrow \nabla (\alpha \wedge \gamma))))$,

(A15) $\nabla ((\alpha \wedge \gamma) \wedge (\beta \wedge \gamma)) \rightarrow \nabla ((\gamma \wedge \alpha) \wedge (\gamma \wedge \beta))$,

(A16) $(\nabla \alpha \rightarrow \nabla (\alpha \wedge \beta)) \rightarrow (\sim (\alpha \wedge \beta) \rightarrow \sim \alpha)$,

(A17) $\sim \beta \rightarrow \sim (\alpha \wedge \beta)$,

(A18) $(\nabla \alpha \rightarrow \nabla (\alpha \wedge \beta)) \rightarrow ((\beta \rightarrow \alpha) \rightarrow (\nabla \sim \alpha \rightarrow \nabla (\sim \alpha \wedge \sim \beta)))$,

(A19) $\nabla \alpha \rightarrow \nabla (\alpha \wedge (\alpha \wedge \sim (\sim \alpha \wedge \sim \beta)))$,

(A20) $\alpha \rightarrow \sim (\sim \alpha \wedge \sim \beta)$,

(A21) $(\alpha \Rightarrow \beta) \leftrightarrow ((\alpha \rightarrow \beta) \wedge (\nabla \alpha \rightarrow \nabla (\alpha \wedge \beta)))$,

(A22) $(\alpha \Rightarrow \beta) \leftrightarrow (\alpha \rightarrow \beta)$,

(A23) $\sim (\sim (\gamma \wedge \alpha) \wedge \sim (\beta \wedge \alpha)) \leftrightarrow (\alpha \wedge \sim (\sim \beta \wedge \sim \gamma))$, (A24) $\sim(\alpha \wedge \sim\nabla\alpha)$,

(A25) $\sim\alpha \to (\nabla\alpha \to \alpha)$,

(A26) $\nabla\alpha \to \nabla(\alpha \wedge \nabla\alpha)$,

(A27) $\nabla \sim \alpha \to (\nabla\nabla\alpha \to \nabla(\sim\alpha \wedge \alpha))$,

(A28) $(\alpha \Rightarrow \beta) \to ((\beta \Rightarrow \alpha) \to (\nabla(\gamma \Rightarrow \alpha) \to \nabla((\gamma \Rightarrow \alpha) \wedge (\gamma \Rightarrow \beta))))$,

(A29) $(\alpha \Rightarrow \beta) \to ((\beta \Rightarrow \alpha) \to (\nabla(\alpha \Rightarrow \gamma) \to \nabla((\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma))))$,

(A30) $\alpha \to (\nabla\beta \to \nabla(\beta \wedge \alpha))$.

The consequence operation $C_{\mathcal{L}}$ in $\mathcal{L} = (A^0, F)$ is determined by the set $\mathcal{A}_l$ and by the following rule of inference:

(R1) $\dfrac{\alpha, \alpha \to \beta}{\beta}$, (Modus Ponens)

The system $\mathcal{M}_4 = (\mathcal{L}, C_{\mathcal{L}})$ thus obtained, will be called four-valued Monteiro propositional calculus. It is worth mentioning that the above connectives are not independent, however, we consider them for simplicity. We shall denote by $\mathcal{T}$ the set of all formulas derivable in $\mathcal{M}_4$. As usual, if $\alpha \in \mathcal{T}$ we shall say that $\alpha$ is a theorem of $\mathcal{M}_4$ and we shall write $\vdash \alpha$ or $\alpha$.

In Lemma 1.1 we summarize the most important rules and theorems necessary for further development.

**Lemma 1.1.** *In* $\mathcal{M}_4$ *the following rules and theorems hold*:

(R2) $\dfrac{\alpha}{\beta \to \alpha}$,

(R3) $\dfrac{\alpha \to (\beta \to \gamma)}{(\alpha \to \beta) \to (\alpha \to \gamma)}$,

(T1) $\alpha \to \alpha$,

(T2) $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$,

(R4) $\dfrac{\alpha \to \beta}{(\gamma \to \alpha) \to (\gamma \to \beta)}$,

(R5) $\dfrac{(\alpha \to \beta) \to (\alpha \to \gamma)}{\beta \to (\alpha \to \gamma)}$,

(R6) $\dfrac{\alpha \to (\beta \to \gamma)}{\beta \to (\alpha \to \gamma)}$,

(T3) $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$,

(T4) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,

(R7) $\dfrac{\alpha \to \beta, \beta \to \gamma}{\alpha \to \gamma}$,

(R8) $\dfrac{\alpha \to \beta}{(\beta \to \gamma) \to (\alpha \to \gamma)}$,

(R9) $\dfrac{\alpha, \beta}{\alpha \wedge \beta}$, (R10) $\dfrac{\alpha,\alpha \Rightarrow \beta}{\beta}$,

(R11) $\dfrac{\alpha \Rightarrow \beta}{(\alpha \to \beta) \wedge (\nabla \alpha \to \nabla(\alpha \wedge \beta))}$,

(R12) $\dfrac{(\alpha \to \beta) \wedge (\nabla \alpha \to \nabla(\alpha \wedge \beta))}{\alpha \Rightarrow \beta}$,

(T5) $\alpha \Rightarrow \alpha$,

(R13) $\dfrac{\alpha \Rightarrow \beta, \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$,

(R14) $\dfrac{\alpha}{\beta \Rightarrow \alpha}$,

(T6) $(\alpha \to \beta) \to (\nabla(\alpha \wedge \beta) \to (\nabla(\gamma \wedge \alpha) \to \nabla(\beta \wedge \gamma)))$,

(T7) $\nabla(\alpha \wedge \beta) \to \nabla(\beta \wedge \alpha)$,

(T8) $\nabla(\alpha \wedge \beta) \to \nabla \alpha$,

(T9) $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta)))$,

(T10) $\nabla(\alpha \wedge \beta) \to (\nabla \alpha \wedge \nabla \beta)$,

(T11) $\nabla(\alpha \wedge \beta) \to \nabla(\nabla \alpha \wedge \nabla \beta)$,

(R15) $\dfrac{\alpha \Rightarrow \beta}{\nabla \alpha \Rightarrow \nabla \beta}$,

(T12) $(\alpha \wedge \beta) \to (\beta \wedge \alpha)$,

(T13) $(\sim(\alpha \wedge \beta) \to \sim \alpha) \to (\sim \beta \to \sim \alpha)$,

(R16) $\dfrac{\alpha \Rightarrow \beta, \beta \Rightarrow \alpha}{\sim \alpha \Rightarrow \sim \beta}$,

(T13) $(\beta \to \alpha) \to ((\alpha \to \nabla \gamma) \to (\beta \to \nabla \gamma))$,

(R17) $\dfrac{\beta \Rightarrow \alpha, \gamma \Rightarrow \delta}{(\alpha \to \gamma) \Rightarrow (\beta \to \delta)}$,

(R18) $\dfrac{\alpha \Rightarrow \beta, \gamma \Rightarrow \delta}{(\alpha \wedge \gamma) \Rightarrow (\beta \wedge \delta)}$,

(R19) $\dfrac{\alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow \delta, \delta \Rightarrow \gamma}{(\alpha \Rightarrow \gamma) \Rightarrow (\beta \Rightarrow \delta)}$.

**Proof.** We only prove (R15), (R16) and (R17).
**(R15):**

1) $\alpha \Rightarrow \beta$,

2) $\alpha \to \beta$, [(1), (R11), (A3), (r1)]

3) $\nabla \alpha \to \nabla(\alpha \wedge \beta)$, [(1), (R11), (A4), (r1)]

4) $(\nabla\alpha \to \nabla(\alpha \wedge \beta)) \to (\nabla\alpha \to \nabla\beta)$, [(A7), (R4)]

5) $\nabla\alpha \to \nabla\beta$, [(3), (4), (r1)]

6) $(\nabla\alpha \to (\nabla\alpha \wedge \beta)) \to (\nabla\alpha \to \nabla(\nabla\alpha \wedge \nabla\beta))$, [(T11), (R4)]

7) $\nabla\alpha \to \nabla(\nabla\alpha \wedge \nabla\beta)$, [(3), (6), (r1)]

8) $\nabla\nabla\alpha \to \nabla(\nabla\alpha \wedge \nabla\beta)$, [(A9), (7), (R7)]

9) $\nabla\alpha \Rightarrow \nabla\beta$. [(5), (8), (R9), (R12)]

**(R16):**

1) $\alpha \Rightarrow \beta$,

2) $\beta \Rightarrow \alpha$,

3) $\nabla\alpha \to \nabla(\alpha \wedge \beta)$, [(1), (R11), (A4)]

4) $\beta \to \alpha$, [(2), (R11), (A3)]

5) $\nabla\beta \to \nabla(\beta \wedge \alpha)$, [(2), (R11), (A4)]

6) $\sim(\beta \wedge \alpha) \to \sim\beta$, [(5), (A16), (r1)]

7) $\sim\alpha \to \sim\beta$, [(6), (T13), (r1)]

8) $\nabla\sim\alpha \to \nabla(\sim\alpha \wedge \sim\beta)$, [(A18), (3), (4), (r1)]

9) $\sim\alpha \Rightarrow \sim\beta$. [(7), (8), (R9), (R12)]

**(R17):**

1) $\beta \Rightarrow \alpha$,

2) $\gamma \Rightarrow \delta$,

3) $\beta \to \alpha$, [(1), (R11), (A3)]

4) $\gamma \to \delta$, [(2), (R11), (A3)]

5) $\nabla\gamma \to \nabla(\gamma \wedge \delta)$, [(2), (R11), (A4)]

6) $(\alpha \to \gamma) \to (\beta \to \gamma)$, [(T4), (3), (r1)]

7) $(\alpha \to \nabla\gamma) \to (\beta \to \nabla\gamma)$, [(T14), (3), (r1)]

8) $(\nabla(\alpha \to \gamma) \to \nabla(\alpha \to \gamma)) \to (\nabla(\alpha \to \gamma) \to (\nabla(\beta \to \gamma)) \to (\nabla(\alpha \to \gamma) \to \nabla((\alpha \to \gamma) \wedge (\beta \to \gamma)))$, [(T9)]

9) $(\nabla(\alpha \to \gamma) \to (\nabla(\beta \to \gamma))) \to (\nabla(\alpha \to \gamma) \to \nabla((\alpha \to \gamma) \wedge (\beta \to \gamma)))$, [(8), (T1), (r1)]

10) $(\alpha \to \nabla\gamma) \to \nabla(\beta \to \gamma)$, [(7), (A12), (R7)]

11) $\nabla(\alpha \to \gamma) \to \nabla(\beta \to \gamma)$, [(10), (A11), (R7)]

12) $\nabla(\alpha \to \gamma) \to (\nabla((\alpha \to \gamma) \wedge (\beta \to \gamma)))$, [(11), (9), (r1)]

13) $(\alpha \to \gamma) \Rightarrow (\beta \to \gamma)$, [(6), (12), (R9), (R12)]

14) $(\beta \to \gamma) \to (\beta \to \delta)$, [(T2), (4), (r1)]

15) $(\beta \to \nabla\gamma) \to (\beta \to \nabla(\gamma \wedge \delta))$, [(5), (R4)]

16) $\nabla(\beta \to \gamma) \to (\beta \to \nabla(\gamma \wedge \delta))$, [(A11), (15), (R7)]

17) $(\beta \to \nabla(\gamma \wedge \delta)) \to \nabla(\beta \to (\gamma \wedge \delta))$, [(A12)]

18) $\nabla(\beta \to \gamma) \to \nabla(\beta \to (\gamma \wedge \delta))$, [(16), (17), (R7)]

19) $\nabla(\beta \to \gamma) \to \nabla((\beta \to \gamma) \wedge (\beta \to \delta))$, [(18), (A13), (R7)]

20) $(\beta \to \gamma) \Rightarrow (\beta \to \delta)$, [(14), (19), (R9), (R12)]

21) $(\alpha \to \gamma) \Rightarrow (\beta \to \delta)$. [(13), (20), (R13)] □

**Theorem 1.1.** *The propositional calculus* $\mathcal{M}_4$ *belongs to the class of standard systems of implicative extensional propositional calculi.*

**Proof.** We have to prove that conditions (s1) to (s8) in (Rasiowa, 1974) are satisfied. Clearly, (s1) and (s2) are verified. Besides, (s3), (s4), (s5) and (s6) follow from (T5), (R10), (R13) and (R14), respectively. On the other hand, taking into account (R15) and (R16), we have that (s7) holds. Finally, (R17), (R18) and (R19) allow us to conclude (s8). □

Next, our attention is focused on stating the relationship between four-valued modal algebras and $\mathcal{M}_4$-algebras (Rasiowa, 1974). Lemma 1.2 will be fundamental for this purpose.

**Lemma 1.2.** *In* $\mathcal{M}_4$ *the following theorems hold*:

(T15) $((\alpha \to \beta) \to (\alpha \to \gamma)) \to (\beta \to (\alpha \to \gamma))$,

(T16) $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$,

(T17) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$,

(T18) $((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma))$,

(T19) $\alpha \to (\alpha \wedge \sim(\sim\alpha \wedge \sim\beta))$,

(T20) $\nabla(\alpha \wedge \sim(\sim\alpha \wedge \sim\beta)) \to \nabla((\alpha \wedge \sim(\sim\alpha \wedge \sim\beta)) \wedge \alpha)$.

(T21) $(\alpha \to \beta) \to (\nabla(\alpha \wedge \beta) \to (\nabla(\gamma \wedge \alpha) \to \nabla((\gamma \wedge \alpha) \wedge (\gamma \wedge \beta))))$,

(T22) $(\sim\alpha \wedge \alpha) \to (\sim\alpha \wedge \nabla\alpha)$,

(T23) $(\sim\alpha \wedge \nabla\alpha) \to (\sim\alpha \wedge \alpha)$,

(T24) $\nabla(\sim\alpha \wedge \alpha) \to \nabla(\alpha \wedge \nabla\alpha)$,

(T25) $\nabla(\sim\alpha \wedge \alpha) \to \nabla((\alpha \wedge \nabla\alpha) \wedge (\sim\alpha \wedge \nabla\alpha))$,

(T26) $\nabla(\sim\alpha \wedge \nabla\alpha) \to \nabla(\sim\alpha \wedge \alpha)$,

(T27) $\nabla(\sim\alpha \wedge \nabla\alpha) \to \nabla((\sim\alpha \wedge \nabla\alpha) \wedge (\sim\alpha \wedge \alpha))$.

**Proof.** We only prove (T17), (T20), (T22) and (T27).

**(T17):**

1) $(\alpha \wedge \beta) \to \beta$, [(A4)]

2) $(\beta \to \gamma) \to ((\alpha \wedge \beta) \to \gamma)$, [(1), (R8)]

3) $(\alpha \to (\beta \to \gamma)) \to (\alpha \to ((\alpha \wedge \beta) \to \gamma))$, [(2), (R4)]

4) $(\alpha \to ((\alpha \wedge \beta) \to \gamma)) \to ((\alpha \wedge \beta) \to (\alpha \to \gamma))$, [(T16)]

5) $((\alpha \to (\beta \to \gamma)) \to (\alpha \to ((\alpha \wedge \beta) \to \gamma))) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to (\alpha \to \gamma)))$, [(4), (R4)]

6) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to (\alpha \to \gamma))$, [(3), (5), (r1)]

7) $((\alpha \wedge \beta) \to (\alpha \to \gamma)) \to (((\alpha \wedge \beta) \to \alpha) \to ((\alpha \wedge \beta) \to \gamma))$, [(A2)]

8) $((\alpha \wedge \beta) \to \alpha) \to (((\alpha \wedge \beta) \to (\alpha \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma))$, [(7), (R6)]

9) $((\alpha \wedge \beta) \to (\alpha \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$, [(A3), (8), (r1)]

10) $((\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to (\alpha \to \gamma))) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma))$, [(9), (R4)]

11) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$. [(6), (10), (r1)]

**(T20):**

1) $\nabla \alpha \to \nabla(\alpha \wedge (\alpha \wedge \sim(\sim \alpha \wedge \sim \beta)))$, [(A19)]

2) $(\nabla(\alpha \wedge \sim(\sim \alpha \wedge \sim \beta)) \to \nabla \alpha) \to (\nabla(\alpha \wedge \sim(\sim \alpha \wedge \sim \beta)) \to \nabla(\alpha \wedge (\alpha \wedge \sim(\sim \alpha \wedge \sim \beta))))$, [(1), (R4)]

3) $\nabla(\alpha \wedge \sim(\sim \alpha \wedge \sim \beta)) \to \nabla(\alpha \wedge (\alpha \wedge \sim(\sim \alpha \wedge \sim \beta)))$, [(2), (T8), (r1)]

4) $\nabla(\alpha \wedge (\alpha \wedge \sim(\sim \alpha \wedge \sim \beta))) \to \nabla((\alpha \wedge \sim(\sim \alpha \wedge \sim \beta)) \wedge \alpha)$, [(T7)]

5) $\nabla(\alpha \wedge \sim(\sim \alpha \wedge \sim \beta)) \to \nabla((\alpha \wedge \sim(\sim \alpha \wedge \sim \beta)) \wedge \alpha)$. [(3), (4), (R7)]

**(T22):**

1) $(((\sim \alpha \wedge \alpha) \to \sim \alpha) \wedge ((\sim \alpha \wedge \alpha) \to \nabla \alpha)) \to ((\sim \alpha \wedge \alpha) \to (\sim \alpha \wedge \nabla \alpha))$, [(T18)]

3) $(\sim \alpha \wedge \alpha) \to \sim \alpha$, [(A3)]

4) $(\sim \alpha \to (\sim \alpha \wedge \nabla \alpha)) \to ((\sim \alpha \wedge \alpha) \to \nabla \alpha)$, [(T17)]

5) $\alpha \to \nabla \alpha$, [(A10)]

6) $\sim \alpha \to (\alpha \wedge \nabla \alpha)$, [(4), (R2)]

7) $(\sim \alpha \wedge \alpha) \to \nabla \alpha$, [(3), (5), (r1)]

8) $((\sim \alpha \wedge \alpha) \to \sim \alpha) \wedge ((\sim \alpha \wedge \alpha) \to \nabla \alpha)$, [(2), (6), (R9)]

9) $(\sim \alpha \wedge \alpha) \to (\sim \alpha \wedge \nabla \alpha)$. [(7), (1), (r1)]

**(T27):**

1) $\nabla(\sim \alpha \wedge \nabla \alpha) \to \nabla(\sim \alpha \wedge \alpha)$, [(T26)]

2) $(\nabla(\sim \alpha \wedge \alpha) \to \nabla((\sim \alpha \wedge \nabla \alpha) \wedge (\sim \alpha \wedge \alpha))) \to (\nabla(\sim \alpha \wedge \nabla \alpha) \to \nabla((\sim \alpha \wedge \nabla \alpha) \wedge (\sim \alpha \wedge \alpha)))$, [(1), (R8)]

3) $\nabla((\sim \alpha \wedge \alpha) \wedge (\sim \alpha \wedge \nabla \alpha)) \to \nabla((\sim \alpha \wedge \nabla \alpha) \wedge (\sim \alpha \wedge \alpha))$, [(T7)]

4) $(\nabla(\sim \alpha \wedge \alpha) \to \nabla((\sim \alpha \wedge \alpha) \wedge (\sim \alpha \wedge \nabla \alpha))) \to (\nabla(\sim \alpha \wedge \alpha) \to \nabla((\sim \alpha \wedge \nabla \alpha) \wedge (\sim \alpha \wedge \alpha)))$, [(3), (R4)]

5) $\nabla(\sim\alpha\wedge\alpha)\to\nabla((\sim\alpha\wedge\nabla\alpha)\wedge(\sim\alpha\wedge\alpha))$, [(4), (T25), (r1)]

6) $\nabla(\sim\alpha\wedge\nabla\alpha)\to\nabla((\sim\alpha\wedge\nabla\alpha)\wedge(\sim\alpha\wedge\alpha))$. [(5), (2), (r1)] □

**Remark 1.1.** *It is worth noting that if* $\alpha$ *is a formula derivable in* $\mathcal{M}_4$, *then* $\alpha_{\mathcal{U}}(\upsilon)=1$ *for every valuation* $\upsilon$ *of* $\mathcal{L}$ *in every* $\mathcal{M}_4$*-algebra* $\mathcal{U}$.

**Proposition 1.1.** *Let* $\langle L,\wedge,\vee,\sim,\nabla,1\rangle\in\mathcal{TM}$. *Then* $\langle L,\Rightarrow,\to,\wedge,\sim,\nabla,1\rangle$ *is an* $\mathcal{M}_4$*-algebra, where* $\to$ *and* $\Rightarrow$ *are defined as follows*:

$$x\to y=\nabla\sim x\vee y,$$

$$x\Rightarrow y=(x\to y)\wedge(\nabla x\to\nabla(x\wedge y)).$$

**Proof.** We shall prove that conditions (a1) to (a4) in (Rasiowa, 1974) hold. Indeed, taking into account the definitions of $\to$ and $\Rightarrow$ we have that (a1) and (a2) are satisfied. On the other hand, let $a,b\in L$ be such that $a\Rightarrow b=b\Rightarrow c=1$. Then, we get (1) $a\to b=b\to c=1$ and (2) $\nabla a\to\nabla(a\wedge b)=\nabla b\to\nabla(b\wedge c)=1$. From (1), (A2) and (2), (A14) we have that $a\to c=1$ and $\nabla a\to\nabla(a\wedge c)=1$ respectively. These assertions and the fact that $x\le y$ if and only if $x\to y=1$ and $\nabla x\to\nabla(x\wedge y)=1$ allow us to conclude that $a=c$ and so, (a3) holds. Besides, if $a\Rightarrow b=b\Rightarrow a=1$, then by an analogous argument to that used in the proof of (a3), we get $a=b$, from which we infer (a4). □

**Proposition 1.2.** *Let* $\langle L,\Rightarrow,\to,\wedge,\sim,\nabla,1\rangle$ *be an* $\mathcal{M}_4$*-algebra. Then* $\langle L,\wedge,\vee,\sim,\nabla,1\rangle\in\mathcal{TM}$ *where* $a\vee b=\sim(\sim a\wedge\sim b)$.

**Proof.** To show that $\langle L,\wedge,\vee,\sim,1\rangle$ is a De Morgan algebra, by (Marona, 1964) it suffices to prove that conditions (M1) $a=a\wedge\sim(\sim a\wedge\sim b)$ and (M2) $a\wedge\sim(\sim b\wedge\sim c)=\sim(\sim(c\wedge a)\wedge\sim(b\wedge a))$ are satisfied. By virtue of Remark 1.1 and (a4) in (Rasiowa, 1974), (M2) follows from (A23) and (M1) from (A3), (A19), (T19), (T20), (R9) and (R12). Moreover, following an analogous reasoning and taking into account (A24) we infer (L3); besides, (T22), (T23), (T24), (T27), (R9) and (R12) allow us to conclude (L4). Hence, the proof is complete. □

From Propositions 1.1 and 1.2 we infer:

**Theorem 1.2.** *The notions of* $\mathcal{M}_4$*-algebra and four-valued modal algebra are equivalent.*

Let $\equiv$ be the binary relation on $F$ defined as follows:

$\alpha\equiv\beta$ if and only if $\vdash\alpha\Rightarrow\beta$ and $\vdash\beta\Rightarrow\alpha$ in $\mathcal{M}_4$.

Then, $\equiv$ is a congruence relation on $\langle F,\Rightarrow,\to,\wedge,\sim,\nabla\rangle$ and $\mathcal{T}$ determines an equivalence class which will be denoted by 1. Moreover, $\langle F/\equiv,\Rightarrow,\to,\wedge,\sim,\nabla,1\rangle$ is an $\mathcal{M}_4$-algebra (Rasiowa, 1974) and therefore from Proposition 1.2, we conclude:

**Theorem 1.3.** $F=\langle F/\equiv,\wedge,\sim,\nabla,1\rangle\in\mathcal{TM}$.

On the other hand, since $\mathcal{M}_4$ is consistent, from (Rasiowa, 1974) and Theorem 1.2 we have that the completeness theorem for $\mathcal{M}_4$ holds, which is included in.

**Theorem 1.4.** *Let* $\alpha$ *be a formula of* $\mathcal{M}_4$. *Then the following conditions are equivalent*:

1) $\alpha$ is derivable in $\mathcal{M}_4$;

2) $\alpha$ is valid in every $\mathcal{M}_4$-algebra;

3) $\alpha_{\mathcal{F}}(\upsilon^0)=1$, where $\upsilon^0$ is the canonical valuation in the algebra $\mathcal{F}$.

## 3. Categorical Equivalence between $\mathfrak{TM}$ and $\mathfrak{MDL}_i$

In this section we introduce the notion of modal distributive lattice with implication and we prove some properties of these algebras which allow us to show the announced categorical equivalence.

**Definition 2.1.** *A modal distributive lattice with implication* (*or* $mdl_i$*-algebra*) *is an algebra* $\langle A,\wedge,\vee,\to,\nabla,0,1\rangle$ *of type* $(2,2,2,1,0,0)$ *where* $\langle A,\wedge,\vee,0,1\rangle$ *is a bounded disitributive lattice satisfying the following identities*:

(I1) $x\to x=1$,

(I2) $x\to(y\to z)=(x\to y)\to(x\to z)$,

(I3) $(x\to y)\to x=x$,

(I4) $(x \wedge y) \to z = x \to (y \to z)$,

(I5) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,

(I6) $(x \to y) \wedge y = y$,

(I7) $\nabla 0 = 0$,

(I8) $\nabla(\nabla x \wedge y) = \nabla x \wedge \nabla y$,

(I9) $\nabla(x \to y) = x \to \nabla y$,

(I10) $x \vee (x \to y) = 1$,

(I11) $(\nabla x \to \nabla(x \to y)) \to \nabla((\nabla x \to x) \wedge (\nabla y \to y)) = 1$,

(I12) $(\nabla x \to \nabla(x \to y)) \to x = (\nabla x \to \nabla(x \wedge y)) \to (x \wedge ((x \to y) \to y))$.

In Proposition 2.1 we show some properties of $mdl_i$ -algebras which will be useful in what follows.

**Proposition 2.1.** *Let* $A$ *be an* $mdl_i$ *-algebra. Then the following identities hold*:

(I13) $1 \to x = x$,

(I14) $x \to 1 = 1$,

(I15) $\nabla(\nabla x \to x) = 1$,

(I16) $\nabla 1 = 1$,

(I17) $x \to \nabla x = 1$,

(I18) $\nabla x \to \nabla(x \wedge \nabla x) = 1$,

(I19) $x \to (x \wedge y) = x \to y$,

(I20) $x \to (y \to x) = 1$,

(I21) $x \to (y \to z) = y \to (x \to z)$,

(I22) $(\nabla x \to \nabla(x \wedge y)) \to x = (x \to y) \to ((\nabla x \to \nabla(x \wedge y)) \to (x \wedge y))$,

(I23) $x \wedge \nabla x = x$,

(I24) $x \leq y$ implies $\nabla x \leq \nabla y$,

(I25) $x \leq y$ implies $z \to x \leq z \to y$,

(I26) $x \leq y$ if and only if $x \to y = 1$, $\nabla x \to \nabla(x \wedge y) = 1$,

(I27) $x \leq y$ implies $(y \to z) \to (x \to z) = 1$,

(I28) $((\nabla x \to x) \to x) \to \nabla x = 1$,

(I29) $\nabla\nabla x = \nabla x$.

**Proof.** We only prove (I22) and (I26).

(I22): From (I21), (I5), (I3) and (I12) we have that

$$\begin{aligned}(x \to y) \to ((\nabla x \to \nabla(x \wedge y)) \to (x \wedge y)) &= (\nabla x \to \nabla(x \wedge y)) \to ((x \to y) \to (x \wedge y)) \\ &= ((\nabla x \to \nabla(x \wedge y)) \to (x \wedge ((x \to y) \to y)). \\ &= (\nabla x \to \nabla(x \wedge y)) \to x\end{aligned}$$

(I26): Let $x \le y$. Then, from (I4), (I1), and (I14) we conclude that $x \to y = 1$. Besides, (I1) allows us to infer $\nabla x \to \nabla(x \wedge y) = 1$. Conversely, from the hypothesis, (I13) and (I12) we have that $x = (\nabla x \to \nabla(x \wedge y)) \to x = (\nabla x \to \nabla(x \wedge y)) \to (x \wedge ((x \wedge y) \to y)) = x \wedge y$, and so, $x \le y$. □

Let $\mathcal{A} = \langle A, \wedge, \vee, \to, \nabla, 0, 1 \rangle$ be an $mdl_i$-algebra. Then, we define $\Theta(\mathcal{A}) = \langle A, \wedge, \vee, \sim, \nabla, 0, 1 \rangle$, where $\sim x = (x \to 0) \wedge (\nabla x \to x)$ for every $x \in A$. Furthermore, given the $mdl_i$-algebras $\mathcal{A}$, $\mathcal{A}'$ and a homomorphism $f : \mathcal{A} \to \mathcal{A}'$, we define $\Theta(f) : \Theta(\mathcal{A}) \to \Theta(\mathcal{A}')$ by $\Theta(f) = f$.

Lemma 2.1 and Propositions 2.2, 2.3 and 2.4 are fundamental in order to prove Theorem 2.1.

**Lemma 2.1.** *Let* $A$ *be an* $mdl_i$*-algebra. Then the following properties are verified*:

(I30) $\nabla x \wedge (\nabla x \to x) = x$,

(I31) $(\nabla x \to x) \to x = \nabla x$,

(I32) $((x \to 0) \to 0) \to x = 1$,

(I33) $\nabla(x \to 0) = x \to 0$,

(I34) $((x \to 0) \wedge (\nabla x \to x)) \to 0 = \nabla x$,

(I35) $\nabla \sim x = x \to 0$,

(I36) $x \le y$ implies $(y \to 0) \to (x \to 0) = 1$,

(I37) $x \le y$ implies $(\nabla y \to y) \to (\nabla x \to x) = \nabla x \to (y \to x)$,

(I38) $\nabla(\sim y \wedge \sim x) = (y \to 0) \wedge (x \to 0)$,

(I39) $x \le y$ implies $\sim y \le \sim x$.

**Proof.** We only prove (I30) and (I39).

(I30): From (I5), (I17), (I20) and (I4) we have that $x \to (\nabla x \wedge (\nabla x \to x)) = (x \to \nabla x) \wedge (x \to (\nabla x \to x)) = 1$ and $(\nabla x \wedge (\nabla x \to x)) \to x = 1$. Besides, (I23), (I6) and (I1) allow us to infer that $\nabla x \to \nabla(\nabla x \wedge (\nabla x \to x) \wedge x) = \nabla x \to \nabla(x \to (\nabla x \to x)) = 1$. In addition, from (I8), (I23), (I6) and (I26) we have $\nabla(\nabla x \wedge (\nabla x \to x)) \to \nabla(\nabla x \wedge (\nabla x \to x) \wedge x) = (\nabla x \wedge \nabla(\nabla x \to x)) \to \nabla x = 1$. By these assertions and (I26) we conclude the proof.

(I39): From (I4), (I5) and (I31) we get that

$$\begin{aligned}\sim y \to \sim x &= (y \to 0) \to ((\nabla y \to y) \to ((x \to 0) \wedge (\nabla x \to x)))\\ &= (y \to 0) \to (((\nabla y \to y) \to (x \to 0)) \wedge ((\nabla y \to y) \to (\nabla x \to x)))\\ &= (y \to 0) \to (((\nabla y \to y) \to (x \to 0)) \wedge (\nabla x \to (y \to x)))\\ &= ((y \to 0) \to ((\nabla y \to y) \to (x \to 0))) \wedge ((y \to 0) \to (\nabla x \to (y \to x))).\end{aligned}$$

Then, by (I21), (I36), (I14) and (I2) we conclude that $\sim y \to \sim x = (\nabla x \to (y \to (0 \to x))) = 1$. On the other hand, from (I35), (I38), (I19) and (I36) we have that $\nabla \sim y \to \nabla(\sim y \wedge \sim x) = (y \to 0) \to ((y \to 0) \wedge (x \to 0)) = (y \to 0) \to (x \to 0) = 1$. Hence, the above assertions and (I26) allow us to conclude that $\sim y \le \sim x$. □

**Proposition 2.2.** $\Theta(\mathcal{A})$ *is a four-valued modal algebra*.

**Proof.** We only have to prove that $\Theta(\mathcal{A})$ satisfies conditions (L1)-(L4).

(L1): Taking into account (I34), (I33), (I8) and (I15) we infer that

$$\sim\sim x = \nabla x \wedge ((x \to 0) \to ((x \to 0) \wedge (\nabla x \to x))).$$

then, by (I19), (I21), (I3) and (I30) we have that

$$\sim\sim x = \nabla x \wedge ((x \to 0) \wedge (\nabla x \to x)) = \nabla x \wedge (\nabla x \to ((x \to 0) \to x)) = \nabla x \wedge (\nabla x \to x) = x.$$

(L2): It is a direct consequence of (I39) and (L1).

(L3): From (I23) and (I10) we conclude that $(x \to 0) \vee \nabla x = 1$. This identity and (I10) allow us to infer that $\sim x \vee \nabla x = 1$.

(L4): It follows from (I6) and (I30). □

**Proposition 2.3.** $\Theta$ *is a full and faithful functor between* $mdl_i$ *-algebras and four-valued modal algebras.*

**Proof.** It follows from Proposition 2.2 and the definition of $\Theta$. □

**Proposition 2.4.** *Every four-valued modal algebra* $\mathcal{A} = \langle A, \wedge, \vee, \sim, \nabla, 0, 1 \rangle$ *is isomorphic (equal) to* $\Theta(\mathcal{B})$ *for some* $mdl_i$ *-algebra* $\mathcal{B}$.

**Proof.** Let $\mathcal{B} = \langle A, \wedge, \vee, \to, \nabla, 0, 1 \rangle$, where $x \to y = \nabla \sim x \vee y$ for all $x, y \in A$. As properties (I1)-(I12) hold in $\mathbb{M}_4$, then $\mathcal{B}$ is an $mdl_i$ -algebra. In order to complete the proof, we must show that $\Theta(\mathcal{B}) = \mathcal{A}$. If we define $-x = (x \to 0) \wedge (\nabla x \to x)$, by Proposition 2.2 it follows that $\Theta(\mathcal{B}) = \langle A, \wedge, \vee, -, \nabla, 0, 1 \rangle$ is a four-valued modal algebra. In addition, for every $x \in A$, $\sim x = (x \to 0) \wedge (\nabla x \to x)$, since this identity holds in $\mathbb{M}_4$. □

**Theorem 2.1.** $\Theta$ *is a categorical equivalence between* $\mathfrak{MDL}_i$ *and* $\mathfrak{TM}$.

**Proof.** It is a direct consequence of Propositions 2.3, 2.4 and Theorem 1 of (MacLane, 1971). □

Theorem 2.1 and the results on Section 1 confirm that $mdl_i$ -algebras are the algebraic counterpart of the four-valued Monteiro propositional calculus.

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