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Stability Analysis of the Chaotic Reverse Butterfly-Shaped Dynamical System Represented in State Variable form Using Hurwitz Polynomials

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The stability of a dynamic system of a differential equations in state variable form describes how it responds to significantly small perturbations. This qualitative behavior a of system of differential equations is studied using Lyapunov or Hurwitz polynomials. The latter reduces the problem of stability of equilibrium points of nonlinear systems to an algebraic linearized system, providing necessary and sufficient criteria in terms of

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Hurwitz determinant or Routh - Hurwitz Array for which the system is stable. In this paper, the stability analysis of the chaotic reverse butterfly-shaped dynamical system is presented using Hurwitz polynomials. The proposed procedure has been illustrated lucidly and validated with numerical simulations in MAPLE software.

Keywords: Differential equations; dynamical systems; Hurwitz polynomial; stability; Routh-Hurwitz criterion; bifurcation; chaos; lyapunov exponents; reverse butterfly-shaped system.

2010 Mathematics Subject Classification: 34D10, 34H10, 34H20.

1 Introduction

In the study of systems of differential equations, stability analysis aims at establishing necessary and sufficient conditions for which trajectories close to the system's initial condition remain so at all future times or tend to stationary solutions. Stability theory concerns the qualitative behavior of a dynamic system's response to significantly small perturbation to initial condition [1]. This notion of proximity of variation in the initial conditions of systems began in the early days of the study of mechanics. Initially studied by famous physicists and mathematicians such as Lagrange and Dirichlet, Aleksandr Mikhailovich Lyapunov, a Russian mathematician is known to have laid the foundation of stability theory in his Ph.D. dissertation titled 'The General Problem of Motion Stability' in the year 1892 [2, 3]. Lyapunov established two methods for analyzing the stability of a system of ordinary differential equations, of which one is the based on the use of Hurwitz polynomials.

Chaotic systems are fundamentally sensitive to initial conditions. Henri Poincare had a first glimpse of the complexity chaos in his quest to find a solution to Newton's three body problem during a competition in honor of King Oscar II [4]. Edward Lorenz, a meteorologist and professor at MIT reintroduced the theory of chaos in the year 1960 during his research to simulate and predict the weather condition. Lorenz developed a deterministic model which he would simulate, yet he could not predict the outcome. He observed in his study a significant variation in his climatological results with a slight variation (decimal difference) in the initial condition he entered mistakenly. Lorenz' experience awakened research into stability, chaos, bifurcation among others [5, 6, 7]. The reverse butterfly-shaped dynamical system is a chaotic system similar to the Lorenz system but with different topological structure which was first propsed by [8].

To determine the stability of the nonlinear system expressed in the form $\dot{x} = Ax$, it is relevant to observe the nature of the roots of the characteristic polynomial associated with the corresponding eigenvalues of A. Thus, the problem of stability analysis of equlibrium points of the nonlinear system is reduced to an algebraic linearized system. The system is considered asymptotically stable if all the roots of the characteristics polynomial associated with A lie in the left half of the complex plane. In other words, the roots must have negative real parts. Such a characteristic polynomial is called a Hurwitz polynomial [9].

This paper presents stability analysis of the chaotic reverse butterfly-shaped system using the standard Routh-Hurwitz stability criteria. Some fundamental properties such as senstivity to initial conditions, computation of Lyapunov Exponents are reported. The paper is structured as follows; section one introduces the subject, section two reviews Hurwitz polynomials and Routh-Hurwitz stability criteria. Section three presents a stability analysis of the reverse butterfly-shaped system. Finally, we perform simulation of numerical results using Maple Software in section four.

2 Hurwitz Polynomial

Let

$$\dot{x} = Ax \tag{2.1}$$

be the linearised system of the nonlinear system

$$\dot{x} = f(x) \tag{2.2}$$

where x is a vector and A is a square matrix. The nature of stability of the equilibrium points of the nonlinear system (2.2) can be determined from the nature of eigenvalues of the associated matrix A of (2.1). If all the eigenvalues of A are negative, then we conclude that (2.2) is asymptotically stable. Thus, the problem of determining stability of the system simply means finding necessary and sufficient conditions for which all roots of the characteristic polynomial lie in the left half of the complex plane.

2.1 Routh – Hurwitz criterion

Suppose that

$$p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n \tag{2.3}$$

is the characteristic polynomial corresponding to the matrix A of the linear system (2.1). To present the Routh-Hurwitz criterion, we first construct a matrix H from the coefficients of the characteristic polynomial (2.3).

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 \\ 0 & a_2 & a_4 & a_6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \tag{2.4}$$

The matrix is constructed as follows; write the coefficients of the polynomial (2.3) with odd positions starting with a_1 on the first row of H. In the second row are the coefficients of the polynomial (2.3) with even location starting with a_0. Subsequent row and column entries are formed by;

$$h_{ij} = \begin{cases} a_{2j-i}, & 0 < 2j - i \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

As a result of the construction, with the exception of the last element on the leading diagonal of H, which has the coefficient a_n - the last coefficient of the polynomial (2.3), all other entries of the last column of H are null. The matrix H is called Hurwitz matrix.

Theorem 2.1. *The polynomial in equation (2.3), with its positive leading coefficient* (a_0) *is a Hurwitz polynomial if and only if all the diagonal principal minors of the Hurwitz matrix are positive [10].*

The principal diagonal minors of the matrix H are;

$$\Delta_1 = |a_1|, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

$$\Delta_n = a_n \cdot \Delta_{n-1}.$$

Also, from the polynomial in equation (2.3), the Routh array is constructed as follows;

$$\begin{array}{c|cccc} \lambda^n & a_0 & a_2 & a_4 & \cdots \\ \lambda^{n-1} & a_1 & a_3 & a_5 & \cdots \\ \lambda^{n-2} & b_0 & b_1 & b_2 & \cdots \\ \lambda^{n-3} & c_0 & c_1 & c_2 & \cdots \\ \lambda^{n-4} & d_0 & d_1 & d_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{2.5}$$

Again, notice that the first row of the array in equation (2.5) begins with a_0 which is the first coefficient of the polynomial in equation (2.3), followed by coefficients with even locations. In the second row are the coefficients of the polynomial in equation (2.3) with odd locations starting with a_1. Subsequent array entries are obtained as follows;

$$\begin{aligned} b_0 &= a_2 - \tfrac{a_0}{a_1}a_3, \quad b_1 = a_4 - \tfrac{a_2}{a_3}a_5, \quad \cdots \\ c_0 &= a_3 - \tfrac{a_1}{b_0}b_3, \quad c_1 = a_5 - \tfrac{a_3}{b_1}b_2, \quad \cdots \\ d_0 &= b_1 - \tfrac{b_0}{c_0}c_1, \quad d_1 = b_2 - \tfrac{b_1}{c_1}c_2, \quad \cdots \end{aligned} \tag{2.6}$$

The number of roots of the polynomial (2.3) in the right half plane of the complex plane is equal to the number of sign variations in the first column of the Routh array in equation (2.5). Moreover, the characteristic polynomial in equation (2.3) is Hurwitz if and only if when performing Routh's array in (2.5), all the values in the first column are nonzero of the same sign [11].

3 Reverse Butterfly-Shaped System

The reverse butterfly-shaped system is given by

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= bx + kxz \\ \dot{z} &= -cz - hxy \end{aligned} \tag{3.1}$$

where x, y, z are the state variables and a, b, c, h, k are positive parameters. Having just two nonlinear terms xy and xz, the deterministic system exhibits complicated dynamics such as randomness yet without any stochastic input as shown in Fig. 1. There is a trend in the timeseries solution for a very short time after which it is thrown into a state of uncertainty or unpredictability. To demonstrate the system's sensitivity to initial conditions which

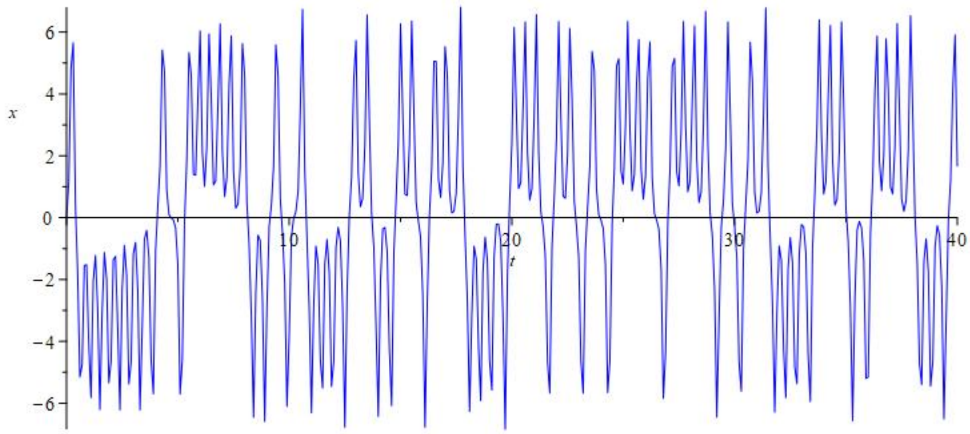

Fig. 1. Timeseries Solution of the System (3.1)

is a fundamental property of chaotic systems, we construct a time series plot for the flow of x for the initial conditions $x(0) = y(0) = 1, z(t) = 0$ captured in blue color and $x(0) = 1.001, y(0) = 1, z(0) = 1$ also captured in green color. The Fig. 2. shows a significant difference in the solution with just a 0.1 percentage change in the initial condition of the variable x. It is easy to observe that the solutions for the two set of initial conditions are same for only a short while. At $t = 10$, the two solutions are on different sides of the graph as seen in Fig. 2.

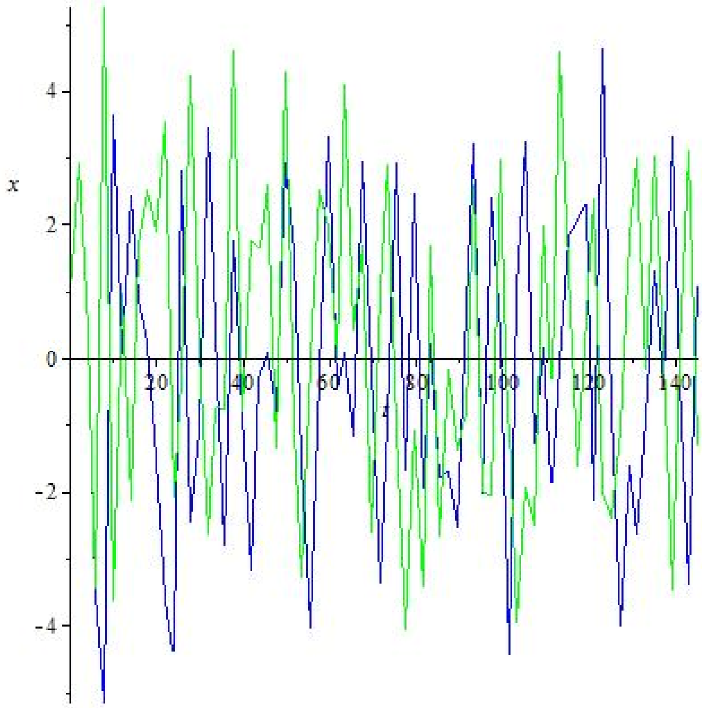

Fig. 2. Sensitivity to Initial Conditions

3.1 Lyapunov Exponents

Lyapunov Exponents (LE) is the average rate of convergence or divergence of trajectories in phase space. It is a measure of the sensitivity of the dynamic system to variation in initial conditions. Fundamentaly, if we take x_0 as initial condition with a nearby point $x_0 + \delta_0$ and let δ_n be the separation of orbit from x_0 and that of $x_0 + \delta_n$. If $|\delta_n| \approx |\delta_0|e^{n\lambda}$, then λ is the Lyapunov Exponent. A positive value of the Lyapunov Exponent confirms the system's sensitivity, indicating that the system is chaotic. A negative value indicates the stability of the dynamic system under consideration. The numerical value of the Lyapunov Exponent indicates the degree of sensitivity of the dynamic system under study. There are as many Lyapunov Exponents as dimensions in the underlying dynamical equations ([12, 13, 14]).

Using Danca's algorithm in [15] for computing Lyapunov Exponents, we obtain the Lyapunov Exponents of the system (3.1) as $LE1 = 1.2674$, $LE2 = -0.0089$ and $LE3 = -13.3502$. LE1 being positive confirms the chaotic behavior of the system (3.1). Thus, close orbits grow exponentially, separating from each other as seen in Fig. 2. Table 1. shows the data for the construction of Fig. 3. generated from Matlab.

3.2 Equilibrium Points

The system (3.1) can be rewritten in the vector form

$$\dot{X} = F$$

where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } F = \begin{pmatrix} a(y-x) \\ bx + kxz \\ -cz - hxy \end{pmatrix}.$$

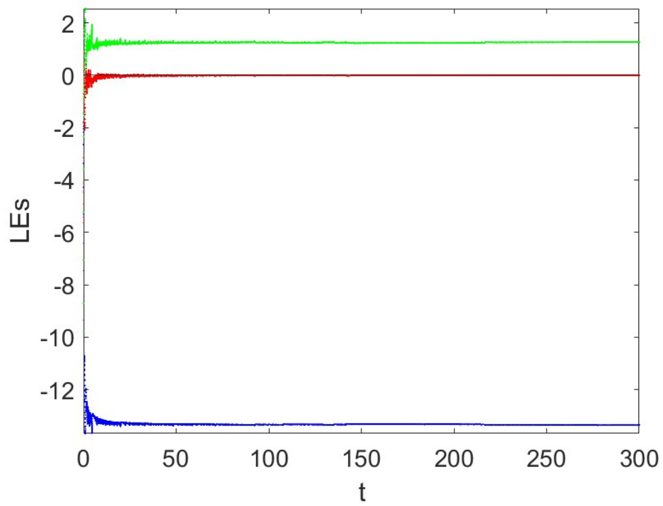

Fig. 3. Lyapunov Exponents for $t \in [0, 300]$

Table 1. Lyapunov exponents for $t \in [0, 300]$

Time	LE1	LE2	LE3
20.00	1.0581	-0.1213	-13.0163
40.00	1.2286	-0.0466	-13.2582
60.00	1.2148	-0.0432	-13.2553
80.00	1.2332	-0.0233	-13.2941
100.00	1.2504	-0.0206	-13.3136
120.00	1.2327	-0.0139	-13.3050
140.00	1.2260	-0.0203	-13.2938
160.00	1.2199	-0.0194	-13.2887
180.00	1.2182	-0.0086	-13.2985
200.00	1.2241	-0.0147	-13.2997
220.00	1.2547	-0.0049	-13.3425
240.00	1.2571	-0.0055	-13.3447
260.00	1.2100	-0.0084	-13.2934
280.00	1.2141	-0.0048	-13.3011
300.00	1.2144	-0.0048	-13.3012

To get the equilibrium points of the system (3.1), we set $F = 0$. Thus,

$$\begin{aligned} a(y-x) &= 0 \\ bx + kxz &= 0 \\ -cz - hxy &= 0 \end{aligned} \tag{3.2}$$

From equation (3.2), we have that the equilibrium points are given at the origin $o(0,0,0)$ and at the points $E^{\pm} = \left(\pm\sqrt{\frac{bc}{kh}}, \pm\sqrt{\frac{bc}{kh}}, -\frac{b}{k}\right)$.

3.3 Stability of the origin $o(0, 0, 0)$

The Jacobian matrix of the system (3.1) is

$$J(x, y, z) = \begin{bmatrix} -a & a & 0 \\ kz + b & 0 & kx \\ -hy & -hx & -c \end{bmatrix} \tag{3.3}$$

Evaluating the Jacobian matrix at the origin $o(0, 0, 0)$ gives

$$J(0, 0, 0) = \begin{bmatrix} -a & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c \end{bmatrix} \tag{3.4}$$

From equation (3.4), we obtain the characteristic polynomial

$$P(\lambda) = \lambda^3 + (a + c)\lambda^2 + (ac - ab)\lambda - abc = 0. \tag{3.5}$$

The Hurwitz matrix associated with the polynomial (3.5) for $a_0 = 1, a_1 = (a + c), a_2 = a(c - b)$ and $a_3 = -abc$ is given by

$$H = \begin{bmatrix} (a + c) & -abc \\ 1 & a(c - b) \end{bmatrix} \tag{3.6}$$

From section (2), the polynomial (3.5) is Hurwitz if and only if

$$\Delta_1 = |a + c| > 0$$

$$\Delta_2 = \begin{vmatrix} (a + c) & -abc \\ 1 & a(c - b) \end{vmatrix} > 0$$

The system's parameters a and c are postive, hence Δ_1 is positive. $\Delta_2 > 0$ if the following condition holds;

$$-a^2b + a^2c + ac^2 > 0. \tag{3.7}$$

The terms a^2c and ac^2 are positive for all positive values of a and c. However, $-a^2b$ is negative because b is a positive parameter. Therefore the characteristic polynomial (3.5) is not Hurwitz.
Also, from the linearized system;

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= bx \\ \dot{z} &= -cz \end{aligned} \tag{3.8}$$

It is immediately observed that, the system's y and z equations break out with solutions $y(t) = e^{bt}$ and $z(t) = e^{-ct}$. This reveals that $y(t)$ grows exponential fast from the origin whereas $z(t)$ approaches the origin exponentially indicating that the system is unstable. Furthermore, judging from the eigenvalues of the charactertic polynomial (3.5), which are;

$$\begin{aligned} \lambda_1 &= -c \\ \lambda_2 &= -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 + 4ab} \\ \lambda_3 &= -\frac{1}{2}a - \frac{1}{2}\sqrt{a^2 + 4ab} \end{aligned} \tag{3.9}$$

From the eigenvalues (3.9), clearly $\lambda_1, \lambda_3 < 0$. However λ_2 is positive because the square root is an increasing function in $[0, \infty)$. We conclude that the origin is a saddle.

3.4 Stability of $E^{\pm}$

Evaluating the Jacobian matrix (3.3) at the equilibria $E^{\pm}$ yields the common characteristic polynomial

$$P^{E^{\pm}}_{a,b,c}(\lambda) = \lambda^3 + (a+c)\lambda^2 + (ac+bc)\lambda + 2abc \tag{3.10}$$

As in the previous case, the matrix H is

$$H = \begin{bmatrix} (a+c) & 2abc \\ 1 & c(a+b) \end{bmatrix} \tag{3.11}$$

for $a_0 = 1, a_1 = (a+c), a_2 = a(a+b)$ and $a_3 = 2abc$. The polynomial (3.10) is Hurwitz if and only if

$$\Delta_1 = |a+c| > 0$$

$$\Delta_2 = \begin{vmatrix} (a+c) & 2abc \\ 1 & c(a+b) \end{vmatrix} > 0.$$

We construct the associated Routh array as follows;

$$\begin{array}{ccc} \lambda^3 & 1 & c(a+b) \\ \lambda^2 & a+c & 2abc \\ \lambda^1 & b_0 & b_1 \\ \lambda^0 & c_0 & c_1 \end{array}$$

where;

$$b_0 = \frac{c(a+b)(a+c) - 2abc}{a+c},$$

and

$$c_0 = 2abc > 0$$

Also, b_1 and c_1 are zero.

From Routh - Hurwitz criterion, the number of roots of the characteristic polynomial with positive real parts is equal to the number of changes in sign of the first column of the Routh array. To have all the roots of the characteristic polynomial on the left side of the complex plane, we should not have any sign variation in the first column of the Routh Array. Since c_0 is positive, it only remains to analyze the sign of b_0 as a function of the variable parameter c (the adjustable parameter of the system (3.1)). We note that if b_0 is positive then the polynomial (3.5) is Hurwitz and the equilibria $E^{\pm}$ are stable. That is, we solve the inequality,

$$\frac{c(a+b)(a+c) - 2bc}{a+c} > 0$$

which yields the condition,

$$c > \frac{a(b-a)}{a+b}. \tag{3.12}$$

Hence, the polynomial (3.5) is Hurwitz if the condition in the inequality (3.12) is satisfied. This implies that the equilibria $E^{\pm}$ of the system (3.1) are asymptotically stable if $c > a(b-a)/(a+b)$.

To ascertain the nature of roots, we construct a graph of the characteristic polynomial using the parameter values $a = 10, b = 40, c = 2.5, k = 16$, and $h = 1$. The characteristic polynomial $P^{E^{\pm}}_{a,b,c}(\lambda)$ crosses the λ-axis just once, indicating the presence of complex eigenvalues as shown in Fig. 4. Next, we investigate the possibility of $P^{E^{\pm}}_{a,b,c}(\lambda)$ having purely imaginary roots. To do this, we take $\lambda = i\omega$ as an eigenvalue and substitute into $P^{E^{\pm}}_{a,b,c}(\lambda)$. We get

$$(i\omega)^3 + (a+c)(i\omega)^2 + (ac+bc)i\omega + 2abc = 0 \tag{3.13}$$

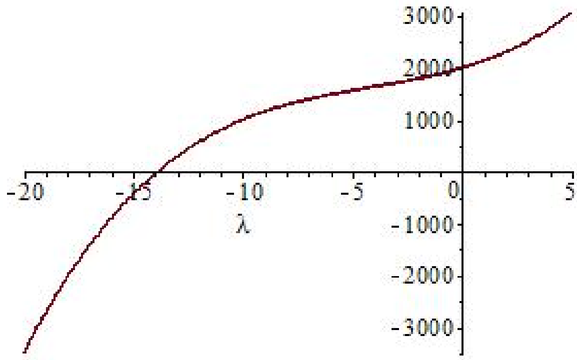

Fig. 4. Graph of $P_{a,b,c}^{E^{\pm}}(\lambda)$

Equating the real and imaginary parts to zero gives

$$c = \frac{2ab}{a+b} - a. \tag{3.14}$$

At the value of c in the equation (3.14), the characteristic polynomial $P_{a,b,c}^{E^{\pm}}(\lambda)$ has purely imaginary roots.

4 Numerical Results

In this section, we verify the stability results obtained from the previous section with simulations performed in the MAPLE software. The parameter c is taken as the adjustable or control parameter for our simulations.

4.1 Stability of the origin $o(0,0,0)$ and $E^{\pm}$

From our analysis, we obtained that the origin $o(0,0,0)$ is unstable for all positive values of c. To demonstrate this, we observe the flow of the system (3.1) close to the origin. In particular, we choose the initial conditions $x(0) = y(0) = z(0) = 0.1$. The trajectory moves away from the origin as time increases indicating that the origin is not attracting. This is shown in the Figs. 5 and 6.

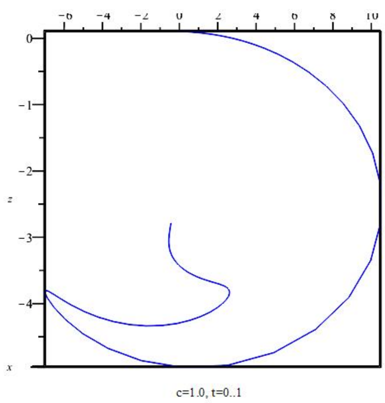

Fig. 5. Stability of the Origin $o(0,0,0)$ at $c = 1$, $t = 0..1$

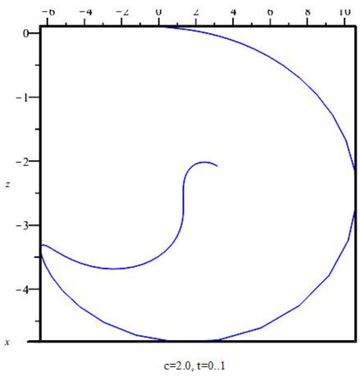

Fig. 6. Stability of the Origin $o(0, 0, 0)$ $c = 2$**,** $t = 0..1$

However, as time increases the trajectory from the unstable manifolds of the origin are attracted onto the stable manifolds of the equilibria $E^{\pm}$ implying that the equilibria $E^{\pm}$ are asymptotically stable as shown in Figs. 7 and 8.

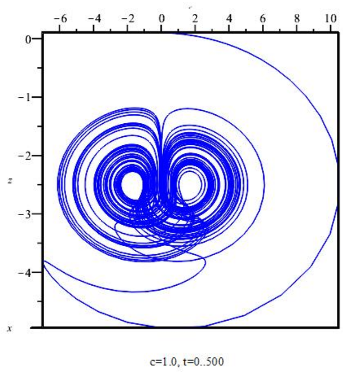

Fig. 7. Stablity of $E^{\pm}$ **at** $c = 1$**,** $t = 0..1$

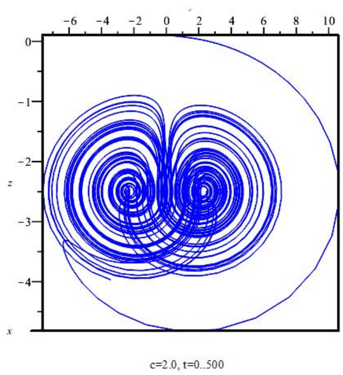

Fig. 8. Stability of $E^{\pm}$ **at** $c = 1$**,** $t = 0..500$

At $c = 2.4$, the equilibria $E^{\pm}$ are chaotic and have strange attractors. At this point, the system is sensitive to any change or disturbance. The trajectory oscillates around one of the equilibria $E^{\pm}$ for a while and jumps onto the other to do the same as shown in Fig. 9. This random behavior is repeated for some time.

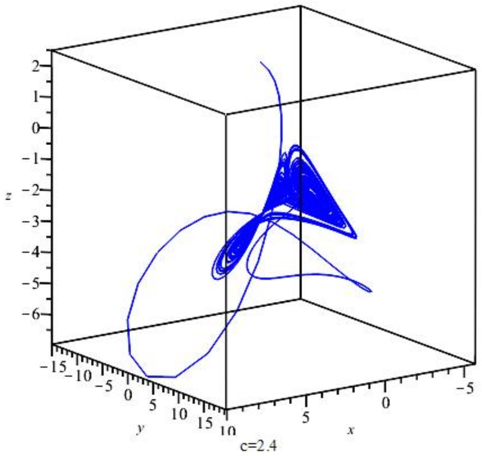

Fig. 9. Chaotic attractors at $c = 2.4$

At $c = 6.0$, the system (3.1) experiences a Hopf bifurcation. This is where the eigenvalues of the equilibria $E^{\pm}$ are non-hyperboic (zero real parts) as computed in equation (3.14). This is characterised by a center manifold as shown in Fig. 10.

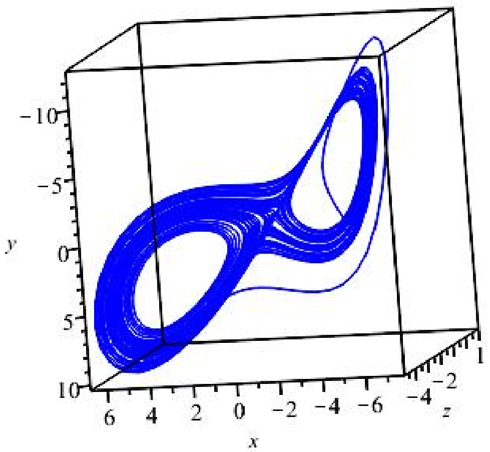

Fig. 10. Bifurcation at $c = 6.0$

At $c > 6.0$, the equilbria $E^{\pm}$ of the system (3.8) have one negative real eigenvalue and a pair of complex conjugates with negative real parts. Fig. 11. shows the phase portrait of the case $c = 7.0$.

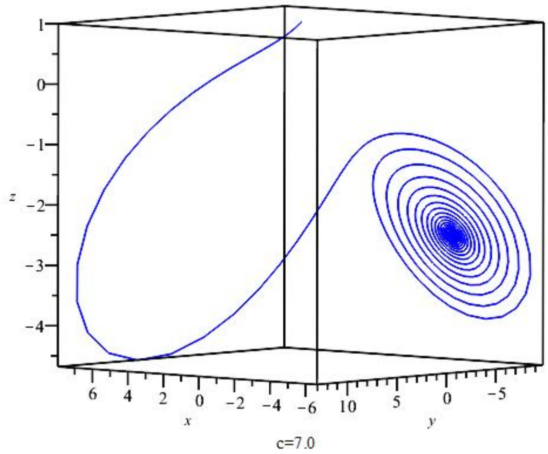

Fig. 11. Stability of $E^{\pm}$ at $c = 7.0$

5 Conclusion

Hurwitz polynomials reduce the problem of stability of equilibrium points of nonlinear systems into an algebraic linearized system providing necessary and sufficient criteria in terms of Hurwitz determinant or Routh - Hurwitz array for which the system is stable. The stability analysis of the chaotic reverse butterfly-shaped dynamical system has been performed using Hurwitz polynomials. The proposed method has been illustrated clearly and validated with numerical simulations in MAPLE software.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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