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Determinants and Inverses of Symmetric Poeplitz and Qoeplitz Matrix$^\sharp$

Jinyu Chen[1*]

[1]*School of Information Science and Engineering, University of Jinan, Jinan 250014, P. R. China.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, we define symmetric Poeplitz and Qoeplitz matrices and give explicit formulae for the determinants and inverses of these matrices by constructing the transformation matrices.

Keywords: Symmetric Poeplitz matrix; symmetric Qoeplitz matrix; Pell sequence; Pell-Lucas sequence; persymmetric Hankel matrix; determinant; inverse.

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1 Introduction

The Pell sequences $\{P_n\}$ and Pell-Lucas sequences $\{Q_n\}$ are defined according to the following recurrence relations [1, 2, 3, 4, 5], And the result are shown below:

$$P_{n+1} = 2P_n + P_{n-1}, \quad n \geq 1,$$
$$Q_{n+1} = 2Q_n + Q_{n-1}, \quad n \geq 1,$$

the initial condition (1)$P_0 = 0$, $P_1 = 1$ (2) $Q_0 = 2$, $Q_1 = 2$.

Corresponding author: E-mail: 377147735@qq.com;

Some scholars showed the explicit determinants and inverses of the special matrices with famous numbers. Sun and Jiang [6] calculated determinant and inverses of the complex Fibonacci Hermitian Toeplitz matrix by constructing the transformation matrices. Determinants and inverses of Fibonacci and Lucas skew symmetric Toeplitz matrices are work out by constructing the special transformation matrices in [7]. In [8], the determinants and inverses are used to be discussed and evaluated Tribonacci skew circulant type matrices. In [9], circulant type matrices with the k-Fibonacci and k-Lucas numbers are considered.What's more, the explicit determinants and inverse matrices are presented by constructing the transformation matrices. It should be noted that Jiang and Zhou [10] obtained the explicit formula about spectral norm of an r-circulant matrix whose entries in the first row are alternately positive and negative, and the authors [11] investigated explicit formulas of spectral norms for g-circulant matrices with Fibonacci and Lucas numbers. The authors [12] proposed the invertibility of the generalized Lucas skew circulant type matrices and assigned their determinants and the inverse matrices. Jiang et al. [13] showed us the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and offered the determinants and the inverses of these matrices. In [14], Jiang and Hong gave the exact determinants of the row skew first-plus-last right circulant matrices and the row skew last-plus-first left circulant matrices involving Padovan, Perrin, Tribonacci and the generalized Lucas numbers by the inverse factorization of polynomial.

In this paper, we use the following convention $0^0 = 1$, we define four special matrices as follows.

Definition 1.1. A symmetric Poeplitz matrix is a square matrix of the form

$$\mathbf{T}_{P,n} = \begin{pmatrix} P_1 & P_2 & P_3 & \cdots & P_{n-2} & P_{n-1} & P_n \\ P_2 & P_1 & P_2 & \ddots & P_{n-3} & P_{n-2} & P_{n-1} \\ P_3 & P_2 & P_1 & \ddots & \ddots & P_{n-3} & P_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ P_{n-2} & P_{n-3} & \ddots & \ddots & P_1 & P_2 & P_3 \\ P_{n-1} & P_{n-2} & P_{n-3} & \ddots & P_2 & P_1 & P_2 \\ P_n & P_{n-1} & P_{n-2} & \cdots & P_3 & P_2 & P_1 \end{pmatrix}_{n\times n}, \tag{1.1}$$

where P_i $(1 \leqslant i \leqslant n)$ are the Pell numbers. It is evidently determined by its first row.

Definition 1.2. A persymmetric Qankel matrix is a square matrix of the form

$$\mathbf{H}_{Q,n} = \begin{pmatrix} Q_n & Q_{n-1} & Q_{n-2} & \cdots & Q_3 & Q_2 & Q_1 \\ Q_{n-1} & Q_{n-2} & Q_{n-3} & \iddots & Q_2 & Q_1 & Q_2 \\ Q_{n-2} & Q_{n-3} & \iddots & \iddots & Q_1 & Q_2 & Q_3 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ Q_3 & Q_2 & Q_1 & \iddots & \iddots & Q_{n-3} & Q_{n-2} \\ Q_2 & Q_1 & Q_2 & \iddots & Q_{n-3} & Q_{n-2} & Q_{n-1} \\ Q_1 & Q_2 & Q_3 & \cdots & Q_{n-2} & Q_{n-1} & Q_n \end{pmatrix}_{n\times n}, \tag{1.2}$$

where $Q_i (1 \leqslant i \leqslant n)$ are the Pell-Lucas numbers. It is evidently determined by its first row.

It is easy to check that

$$\mathbf{H}_{P,n} = \mathbf{T}_{P,n}\hat{I}_n \ , \tag{1.3}$$

$$\mathbf{H}_{Q,n} = \mathbf{T}_{Q,n}\hat{I}_n \ , \tag{1.4}$$

where $\hat{I}_n$ is the "reverse unit matrix", having ones along the secondary diagonal and zeros elsewhere.

2 Determinant and Inverse of the Symmetric Poeplitz Matrix

In this section, we give the determinant and inverse of the matrix $\mathbf{T}_{P,n}$.

Theorem 2.1. *Let* $\mathbf{T}_{P,n}$ *be a symmetric Poeplitz matrix as the form of* (1.1). *Then we have*

$$\det \mathbf{T}_{P,1} = 1, \ \det \mathbf{T}_{P,2} = -3, \ \det \mathbf{T}_{P,3} = 8, \ \det \mathbf{T}_{P,4} = -20$$

and

$$\det \mathbf{T}_{P,n} = (-1)^{n-1}2^{n-3}[K_2(P_{n-2} - P_{n-1}P_2) - K_3(P_{n-1} - P_nP_2)], \ (n > 5), \tag{2.1}$$

where

$$K_2 = \sum_{k=3}^{n}(P_{n+1-k} - P_nP_k)\Delta_{n-k}, \ K_3 = \sum_{k=3}^{n-1}(P_{n-k} - P_{n-1}P_k)\Delta_{n-k} + P_2 - P_{n-1}P_n,$$

$$\Delta_0 = 1, \ \Delta_i = -2(\sum_{k=1}^{i} P_{k+1}\Delta_{i-k}), \ (1 \leqslant i \leqslant n-3).$$

Proof. Let $\mathbf{T}_{P,n}$ be a symmetric Poeplitz matrix of order n, and we can easily get the following conclusions:

$$\det \mathbf{T}_{P,1} = 1, \ \det \mathbf{T}_{P,2} = -3, \ \det \mathbf{T}_{P,3} = 8, \ \det \mathbf{T}_{P,4} = -20.$$

We can introduce the following two transformation matrices when $n > 5$,

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -P_n & \vdots & & & & & \iddots & 1 \\ -P_{n-1} & \vdots & & & & \iddots & 1 & 0 \\ 0 & \vdots & & & \iddots & 1 & 2 & -1 \\ 0 & \vdots & & \iddots & 1 & 2 & -1 & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & 0 & 1 & 2 & -1 & 0 & \iddots & \vdots \\ 0 & 1 & 2 & -1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

and

$$\mathcal{N}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \Delta_{n-3} & \vdots & & & \iddots & 1 & 0 \\ \vdots & \Delta_{n-4} & \vdots & & \iddots & 1 & \iddots & \vdots \\ \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \Delta_2 & 0 & \iddots & \iddots & & & \vdots \\ \vdots & \Delta_1 & 1 & \iddots & & & & \vdots \\ 0 & \Delta_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n},$$

where

$$\Delta_0 = 1,\ \Delta_i = -2(\sum_{k=1}^{i} P_{k+1}\Delta_{i-k}),\ (1 \leqslant i \leqslant n-3).$$

By using $\mathcal{M}_1,\ \mathcal{N}_1$ and the recurrence relations of the Pell sequences, the matrix $\mathbf{T}_{P,n}$ is changed into the following form,

$$\mathcal{M}_1\mathbf{T}_{P,n}\mathcal{N}_1 = \begin{pmatrix} 1 & K_1 & P_{n-1} & P_{n-2} & P_{n-3} & \cdots & P_5 & P_4 & P_3 & P_2 \\ 0 & K_2 & x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_5 & x_4 & x_3 & x_2 \\ \vdots & K_3 & y_{n-1} & y_{n-2} & y_{n-3} & \cdots & y_5 & y_4 & y_3 & y_2 \\ \vdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 4P_2 & 2 & \ddots & & & & & \vdots \\ \vdots & \vdots & 4P_3 & 4P_2 & 2 & \ddots & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 4P_{n-3} & \cdots & \cdots & \cdots & 4P_3 & 4P_2 & 2 & 0 \end{pmatrix}_{n\times n},$$

where

$$\begin{aligned} & x_i = P_{n+1-i} - P_nP_i,\ (2 \leqslant i \leqslant n), \\ & y_i = P_{n-i} - P_{n-1}P_i,\ (2 \leqslant i \leqslant n-1),\ y_n = P_2 - P_{n-1}P_n, \\ & K_1 = \sum_{k=3}^{n} P_k\Delta_{n-k}, \\ & K_2 = \sum_{k=3}^{n}(P_{n+1-k} - P_nP_k)\Delta_{n-k}, \\ & K_3 = \sum_{k=3}^{n-1}(P_{n-k} - P_{n-1}P_k)\Delta_{n-k} + P_2 - P_{n-1}P_n. \end{aligned}$$

By using the Laplace expansion of matrix $\mathcal{M}_1\mathbf{T}_{P,n}\mathcal{N}_1$ along the first column, we can get that

$$\det \mathcal{M}_1\mathbf{T}_{P,n}\mathcal{N}_1 = (-1)^{n-1}2^{n-3}[K_2(P_{n-2} - P_{n-1}P_2) - K_3(P_{n-1} - P_nP_2)].$$

While

$$\det \mathcal{M}_1 = \det \mathcal{N}_1 = (-1)^{\frac{(n-1)(n-2)}{2}},$$

we can obtain $\det \mathbf{T}_{P,n}$ as (2.1), which completes the proof. □

Theorem 2.2. *Let $\mathbf{T}_{P,n}$ be an invertible symmetric Poeplitz matrix and $n > 5$.*

(i) When n is odd, we have

$$
\mathbf{T}_{P,n}^{-1} =
\begin{pmatrix}
\frac{n}{2(n+1)} & \frac{3n-1}{2(n+1)} & \frac{n-2}{n+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{3}{n+1} & \frac{2}{n+1} & \frac{1}{(2n+1)} \\
\frac{3n-1}{2(n+1)} & \frac{10n-6}{2(n+1)} & \frac{9n-15}{2(n+1)} & \ddots & & & & & \frac{12}{n+1} & \frac{8}{n+1} & \frac{2}{n+1} \\
\frac{n-2}{n+1} & \frac{9n-15}{2(n+1)} & \ddots & \ddots & \ddots & & & \iddots & \iddots & \frac{12}{n+1} & \frac{3}{n+1} \\
\vdots & \ddots & \ddots & \frac{n(n+4)-7}{2(n+1)} & \frac{n(n+1)-8}{2(n+1)} & \frac{\frac{n^2+2n-3}{2}}{n+1} & \frac{\frac{n^2-2n+3}{2}}{n+1} & \frac{2(\frac{n-3}{2})^2}{n+1} & \iddots & & \vdots \\
\vdots & & \ddots & \frac{n(n+1)-8}{2(n+1)} & \frac{n(n+4)-1}{2(n+1)} & \frac{n}{2} & \frac{2(\frac{n-1}{2})^2}{n+1} & \frac{\frac{n^2-2n+3}{2}}{n+1} & & & \vdots \\
\vdots & & & \frac{\frac{n^2+2n-3}{2}}{n+1} & \frac{n}{2} & \frac{n+3}{2} & \frac{n}{2} & \frac{\frac{n^2+2n-3}{2}}{n+1} & & & \vdots \\
\vdots & & & \frac{\frac{n^2-2n+3}{2}}{n+1} & \frac{2(\frac{n-1}{2})^2}{n+1} & \frac{n}{2} & \frac{n(n+4)-1}{2(n+1)} & \frac{n(n+1)-8}{2(n+1)} & \ddots & & \vdots \\
\vdots & & \iddots & \frac{2(\frac{n-3}{2})^2}{n+1} & \frac{\frac{n^2-2n+3}{2}}{n+1} & \frac{\frac{n^2+2n-3}{2}}{n+1} & \frac{n(n+1)-8}{2(n+1)} & \frac{n(n+4)-7}{2(n+1)} & \ddots & \ddots & \vdots \\
\frac{3}{n+1} & \frac{12}{n+1} & \iddots & \iddots & & & \ddots & \ddots & \ddots & \frac{9n-15}{2(n+1)} & \frac{n-2}{n+1} \\
\frac{2}{n+1} & \frac{8}{n+1} & \frac{12}{n+1} & & & & & \ddots & \frac{9n-15}{2(n+1)} & \frac{10n-6}{2(n+1)} & \frac{3n-1}{2(n+1)} \\
\frac{1}{(2n+1)} & \frac{2}{n+1} & \frac{3}{n+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{n-2}{n+1} & \frac{3n-1}{2(n+1)} & \frac{n}{2(n+1)}
\end{pmatrix}_{n\times n}
\tag{2.2}
$$

(ii) When n is even, we have

$$
\mathbf{T}_{P,n}^{-1} =
\begin{pmatrix}
\frac{n}{2(n+1)} & \frac{3n-1}{2(n+1)} & \frac{n-2}{n+1} & \cdots & \cdots & \cdots & \cdots & \frac{3}{n+1} & \frac{2}{n+1} & \frac{1}{(2n+1)} \\
\frac{3n-1}{2(n+1)} & \frac{10n-6}{2(n+1)} & \frac{9n-15}{2(n+1)} & \ddots & & & & \frac{12}{n+1} & \frac{8}{n+1} & \frac{2}{n+1} \\
\frac{n-2}{n+1} & \frac{9n-15}{2(n+1)} & \ddots & \ddots & \ddots & & \iddots & \iddots & \frac{12}{n+1} & \frac{3}{n+1} \\
\vdots & \ddots & \ddots & \frac{n(n+4)-6}{2(n+1)} & \frac{n(n+1)-3}{2(n+1)} & \frac{\frac{n^2-2n}{2}}{n+1} & \frac{2(\frac{n-1}{2})^2}{n+1} & \iddots & & \vdots \\
\vdots & & \ddots & \frac{n(n+1)-3}{2(n+1)} & \frac{n(n+4)-2}{2(n+1)} & \frac{n(n+1)+1}{2(n+1)} & \frac{\frac{n^2-2n}{2}}{n+1} & & & \vdots \\
\vdots & & & \frac{\frac{n^2-2n}{2}}{n+1} & \frac{n(n+1)+1}{2(n+1)} & \frac{n(n+4)-2}{2(n+1)} & \frac{n(n+1)-3}{2(n+1)} & \ddots & & \vdots \\
\vdots & & \iddots & \frac{2(\frac{n-1}{2})^2}{n+1} & \frac{\frac{n^2-2n}{2}}{n+1} & \frac{n(n+1)-3}{2(n+1)} & \frac{n(n+4)-6}{2(n+1)} & \ddots & \ddots & \vdots \\
\frac{3}{n+1} & \frac{12}{n+1} & \iddots & \iddots & & \ddots & \ddots & \ddots & \frac{9n-15}{2(n+1)} & \frac{n-2}{n+1} \\
\frac{2}{n+1} & \frac{8}{n+1} & \frac{12}{n+1} & & & & \ddots & \frac{9n-15}{2(n+1)} & \frac{10n-6}{2(n+1)} & \frac{3n-1}{2(n+1)} \\
\frac{1}{(2n+1)} & \frac{2}{n+1} & \frac{3}{n+1} & \cdots & \cdots & \cdots & \cdots & \frac{n-2}{n+1} & \frac{3n-1}{2(n+1)} & \frac{n}{2(n+1)}
\end{pmatrix}_{n\times n},
\tag{2.3}
$$

We can observe that $\mathbf{T}_{P,n}^{-1}$ is not only a symmetric matrix, but also a symmetric matrix along its secondary diagonal, i.e., persymmetric matrix.

Proof. We can introduce the following two transformation matrices when $n > 5$,

$$\mathcal{M}_2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & -\frac{K_3}{K_2} & \ddots & \ddots & & & & \vdots \\ \vdots & 0 & 0 & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n},$$

$$\mathcal{N}_2 = \begin{pmatrix} 1 & -K_1 & C_{n-1} & C_{n-2} & \cdots & C_3 & C_2 \\ 0 & 1 & -\frac{x_{n-1}}{K_2} & -\frac{x_{n-2}}{K_2} & \cdots & -\frac{x_3}{K_2} & -\frac{x_2}{K_2} \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n},$$

where

$$C_i = \frac{K_1 x_i}{K_2} - P_i, \ (2 \leqslant i \leqslant n-1),$$

$x_i (2 \leqslant i \leqslant n), \ K_1, \ K_2, \ K_3$ as in Theorem 2.1.

If we multiply $\mathcal{M}_1 \mathbf{T}_{P,n} \mathcal{N}_1$ by $\mathcal{M}_2$ and $\mathcal{N}_2$, the $\mathcal{M}_1$ and $\mathcal{N}_1$ are as in the proof of Theorem 2.1, so we obtain

$$\mathcal{M}_2 \mathcal{M}_1 \mathbf{T}_{P,n} \mathcal{N}_1 \mathcal{N}_2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & K_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & w_{n-1} & w_{n-2} & w_{n-3} & \cdots & w_5 & w_4 & w_3 & w_2 \\ \vdots & \vdots & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & 4P_2 & 2 & \ddots & & & & & \vdots \\ \vdots & \vdots & 4P_3 & 4P_2 & \ddots & \ddots & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 4P_{n-3} & \cdots & \cdots & \cdots & 4P_3 & 4P_2 & 2 & 0 \end{pmatrix}_{n\times n},$$

where

$$w_i = -\frac{K_3 x_i}{K_2} + y_i, \ (2 \leqslant i \leqslant n-1),$$

$y_i (2 \leqslant i \leqslant n)$ as in Theorem 2.1.

we have

$$\mathcal{M}_2\mathcal{M}_1\mathbf{T}_{P,n}\mathcal{N}_1\mathcal{N}_2 = \phi \oplus \Lambda,$$

where $\phi = \begin{pmatrix} 1 & 0 \\ 0 & K_2 \end{pmatrix}_{2\times 2}$ is a diagonal matrix, and Λ is a Toeplitz-like matrix

$$\Lambda = \begin{pmatrix} w_{n-1} & w_{n-2} & w_{n-3} & \cdots & w_5 & w_4 & w_3 & w_2 \\ 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 4P_2 & 2 & \ddots & & & & & \vdots \\ 4P_3 & 4P_2 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 4P_{n-3} & \cdots & \cdots & \cdots & 4P_3 & 4P_2 & 2 & 0 \end{pmatrix}_{(n-2)\times(n-2)}.$$

$\phi \oplus \Lambda$ is the direct sum of ϕ and Λ. Let $\mathcal{M} = \mathcal{M}_2\mathcal{M}_1$, $\mathcal{N} = \mathcal{N}_1\mathcal{N}_2$, then we obtain

$$\mathbf{T}_{P,n}^{-1} = \mathcal{N}(\phi^{-1} \oplus \Lambda^{-1})\mathcal{M}, \tag{2.4}$$

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -P_n & \vdots & & & & & & & \iddots & 1 \\ \frac{K_3}{K_2}P_n - P_{n-1} & \vdots & & & & & & \iddots & 1 & -\frac{K_3}{K_2} \\ 0 & \vdots & & & & & \iddots & 1 & 2 & -1 \\ \vdots & \vdots & & & & \iddots & 1 & 2 & -1 & 0 \\ \vdots & \vdots & & & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & & & \vdots \\ \vdots & 0 & \iddots & \iddots & \iddots & \iddots & & & & \vdots \\ 0 & 1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}, \tag{2.5}$$

$$\mathcal{N} = \begin{pmatrix} 1 & -K_1 & C_{n-1} & C_{n-2} & \cdots & C_3 & C_2 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \Delta_{n-3} & a_{n-3,n-1} & a_{n-3,n-2} & \cdots & a_{n-3,3} & a_{n-3,2} \\ \vdots & \Delta_{n-4} & a_{n-4,n-1} & a_{n-4,n-2} & \cdots & a_{n-4,3} & a_{n-4,2} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \Delta_1 & a_{1,n-1} & a_{1,n-2} & \cdots & a_{1,3} & a_{1,2} \\ 0 & \Delta_0 & a_{0,n-1} & a_{0,n-2} & \cdots & a_{0,3} & a_{0,2} \end{pmatrix}_{n\times n}, \tag{2.6}$$

where

$$a_{i,j} = \begin{cases} -\dfrac{\Delta_i x_j}{K_2}, & i+j \neq n, \\ 1 - \dfrac{\Delta_i x_j}{K_2}, & i+j = n, \end{cases} \quad (0 \leqslant i \leqslant n-3,\ 2 \leqslant j \leqslant n-1),$$

Δ_i $(0 \leqslant i \leqslant n-3)$ as in Theorem 2.1.

We can observe that the inverse matrix of ϕ is $\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{K_2} \end{pmatrix}_{2\times 2}$.

According to Lemma 2.3 in [15], we have

$$\Lambda^{-1} = \begin{pmatrix} 0 & b_1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & b_2 & b_1 & \ddots & & & \vdots \\ \vdots & b_3 & b_2 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & b_2 & b_1 & 0 \\ 0 & b_{n-3} & \cdots & \cdots & b_3 & b_2 & b_1 \\ B_1 & B_2 & B_3 & \cdots & B_{n-4} & B_{n-3} & B_{n-2} \end{pmatrix}_{(n-2)\times(n-2)},$$

where

$$b_1 = \frac{1}{2},$$
$$b_j = (-1)^{j-1} 2^{-j} \left(\sum_{t_1+2t_2+\cdots+(j-1)t_{j-1}=j-1} \frac{(t_1+\cdots+t_{j-1})!}{t_1!\cdots t_{j-1}!} (-2)^{(j-1)-t_1-\cdots-t_{j-1}} (4P_2)^{t_1} \cdots (4P_j)^{t_{j-1}} \right),$$
$$(2 \leqslant j \leqslant n-3),$$
$$B_1 = \frac{1}{w_2},$$
$$B_j = \frac{(-1)^{2j-1} \det \nabla_{n-1-j}([w_k]_{k=3}^{n-j+1}, 2, 4P_2, \cdots, 4P_{n-1-j})}{2^{n-1-j} w_2},\ (2 \leqslant j \leqslant n-2),$$

with

$$\det \nabla_1(w_3) = w_3,$$
$$\det \nabla_{n-1-j}([w_k]_{k=3}^{n-j+1}, 2, 4P_2, \cdots, 4P_{n-1-j}) = 2^{n-2-j} w_{n-j+1} + \sum_{p=1}^{n-2-j} (-1)^{2+p} 2^{n-j-2-p} w_{n-j-p+1}$$
$$\cdot \left(\sum_{t_1+2t_2+\cdots+pt_p=p} \frac{(t_1+\cdots+t_p)!}{t_1!\cdots t_p!} (-2)^{p-t_1-\cdots-t_p} (4P_2)^{t_1} \cdots (4P_{p+1})^{t_p} \right),\ (2 \leqslant j \leqslant n-3).$$

We obtain that

$$\phi^{-1} \oplus \Lambda^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{1}{K_2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 0 & b_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & b_2 & b_1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & b_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & b_{n-3} & \cdots & \cdots & b_2 & b_1 \\ 0 & 0 & B_1 & B_2 & B_3 & \cdots & B_{n-3} & B_{n-2} \end{pmatrix}_{n\times n}. \tag{2.7}$$

According to formulas (2.4) (2.5) (2.6) (2.7) and the recurrence relations of the Pell sequence, when n is odd, we obtain formula (**??**); when n is even, we obtain formula (2.3). Which completes the proof. □

3 Determinant and Inverse of the Symmetric Qoeplitz Matrix

In this section, we give the determinant and inverse of the matrix $\mathbf{T}_{Q,n}$.

Theorem 3.1. *Let $\mathbf{T}_{Q,n}$ be a symmetric Qoeplitz matrix as the form of* (1.2)*. Then we have*

$$\det \mathbf{T}_{Q,1} = 2,\ \det \mathbf{T}_{Q,2} = -32,\ \det \mathbf{T}_{Q,3} = 480,\ \det \mathbf{T}_{Q,4} = -7186$$

and

$$\det \mathbf{T}_{Q,n} = (-1)^{n-1} 2 \times 4^{n-3}[\hat{K}_2(Q_{n-2} - Q_{n-1}Q_2) - \hat{K}_3(Q_{n-1} - Q_nQ_2)],\ (n > 5), \tag{3.1}$$

where

$$\hat{K}_2 = \sum_{k=3}^{n}(Q_{n+1-k} - Q_nQ_k)\hat{\Delta}_{n-k},\ \hat{K}_3 = \sum_{k=3}^{n-1}(Q_{n-k} - Q_{n-1}Q_k)\hat{\Delta}_{n-k} + Q_2 - Q_{n-1}Q_n,$$

$$\hat{\Delta}_0 = 1,\ \hat{\Delta}_i = -(\sum_{k=1}^{i} P_{k+1}\hat{\Delta}_{i-k}),\ (1 \leqslant i \leqslant n-3).$$

Proof. Let $\mathbf{T}_{Q,n}$ be a symmetric Qoeplitz matrix of order n, and we can easily get the following conclusions:

$$\det \mathbf{T}_{Q,1} = 2,\ \det \mathbf{T}_{Q,2} = -32,\ \det \mathbf{T}_{Q,3} = 480,\ \det \mathbf{T}_{Q,4} = -7186.$$

We can introduce the following two transformation matrices when $n \geqslant 5$,

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{-Q_n}{2} & \vdots & & & & & \iddots & 1 \\ \frac{-Q_{n-1}}{2} & \vdots & & & & \iddots & 1 & 0 \\ 0 & \vdots & & & \iddots & 1 & 2 & -1 \\ 0 & \vdots & & \iddots & 1 & 2 & -1 & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & 0 & 1 & 2 & -1 & 0 & \iddots & \vdots \\ 0 & 1 & 2 & -1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n}$$

and

$$\mathcal{N}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \hat{\Delta}_{n-3} & \vdots & & & \iddots & 1 & 0 \\ \vdots & \hat{\Delta}_{n-4} & \vdots & & \iddots & 1 & \iddots & \vdots \\ \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \hat{\Delta}_2 & 0 & \iddots & \iddots & & & \vdots \\ \vdots & \hat{\Delta}_1 & 1 & \iddots & & & & \vdots \\ 0 & \hat{\Delta}_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n \times n},$$

where

$$\hat{\Delta}_0 = 1,\ \hat{\Delta}_i = -(\sum_{k=1}^{i} P_{k+1} \Delta_{i-k}),\ (1 \leqslant i \leqslant n-3).$$

By using $\hat{\mathcal{M}}_1,\ \hat{\mathcal{N}}_1$ and the recurrence relations of the Pell sequences, the matrix $\mathbf{T}_{Q,n}$ is changed into the following form,

$$\hat{\mathcal{M}}_1 \mathbf{T}_{Q,n} \hat{\mathcal{N}}_1 = \begin{pmatrix} 1 & \hat{K}_1 & Q_{n-1} & Q_{n-2} & Q_{n-3} & \cdots & Q_5 & Q_4 & Q_3 & Q_2 \\ 0 & \hat{K}_2 & \omega_{n-1} & {}_{n-2} & \omega_{n-3} & \cdots & \omega_5 & \omega_4 & \omega_3 & \omega_2 \\ \vdots & \hat{K}_3 & \nu_{n-1} & \nu_{n-2} & \nu_{n-3} & \cdots & \nu_5 & \nu_4 & \nu_3 & \nu_2 \\ \vdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 4Q_4 & 4 & \ddots & & & & & \vdots \\ \vdots & \vdots & 4Q_3 & 4Q_2 & 4 & \ddots & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 4Q_{n-3} & \cdots & \cdots & \cdots & 4Q_3 & 4Q_2 & 4 & 0 \end{pmatrix}_{n \times n},$$

where

$$
\begin{aligned}
&\omega_i = P_{n+1-i} - \frac{P_n}{2} P_i,\ (2 \leqslant i \leqslant n),\\
&\nu_i = P_{n-i} - \frac{P_{n-1}}{2} P_i,\ (2 \leqslant i \leqslant n-1),\ \nu_n = P_2 - \frac{P_{n-1}}{2} P_n,\\
&\hat{K}_1 = \sum_{k=3}^{n} P_k \hat{\Delta}_{n-k},\\
&\hat{K}_2 = \sum_{k=3}^{n} (P_{n+1-k} - P_n P_k) \hat{\Delta}_{n-k},\\
&\hat{K}_3 = \sum_{k=3}^{n-1} (P_{n-k} - P_{n-1} P_k) \hat{\Delta}_{n-k} + P_2 - P_{n-1} P_n.
\end{aligned}
$$

By using the Laplace expansion of matrix $\hat{\mathcal{M}}_1 \mathbf{T}_{Q,n} \hat{\mathcal{N}}_1$ along the first column, we can get that

$$
\det \hat{\mathcal{M}}_1 \mathbf{T}_{Q,n} \hat{\mathcal{N}}_1 = (-1)^{n-1} 2^{n-3} [\hat{K}_2 (Q_{n-2} - Q_{n-1} Q_2) - \hat{K}_3 (Q_{n-1} - Q_n Q_2)].
$$

While

$$
\det \hat{\mathcal{M}}_1 = \det \hat{\mathcal{N}}_1 = (-1)^{\frac{(n-1)(n-2)}{2}},
$$

we can obtain $\det \mathbf{T}_{Q,n}$ as (3.1), which completes the proof. □

Theorem 3.2. *Let $\mathbf{T}_{Q,n}$ be an invertible symmetric Qoeplitz matrix and $n > 5$. Then we have*

$$
\mathbf{T}_{Q,n}^{-1} = \begin{pmatrix}
\zeta_{1,1} & \zeta_{1,2} & \zeta_{1,3} & \cdots & \zeta_{1,n-2} & \zeta_{1,n-1} & \zeta_{1,n} \\
\zeta_{1,2} & \zeta_{2,2} & \zeta_{2,3} & \iddots & \zeta_{2,n-2} & \zeta_{2,n-1} & \zeta_{1,n-1} \\
\zeta_{1,3} & \zeta_{2,3} & \zeta_{3,3} & \iddots & \ddots & \zeta_{2,n-2} & \zeta_{1,n-2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\zeta_{1,n-2} & \zeta_{2,n-2} & \iddots & \iddots & \zeta_{3,3} & \zeta_{2,3} & \zeta_{1,3} \\
\zeta_{1,n-1} & \zeta_{2,n-1} & \zeta_{2,n-2} & \iddots & \zeta_{2,3} & \zeta_{2,2} & \zeta_{1,2} \\
\zeta_{1,n} & \zeta_{1,n-1} & \zeta_{1,n-2} & \cdots & \zeta_{1,3} & \zeta_{1,2} & \zeta_{1,1}
\end{pmatrix}_{n \times n}, \qquad (3.2)
$$

where

$$
\begin{aligned}
\zeta_{1,1} &= \frac{1}{2} + \frac{\hat{K}_1 Q_n}{4\hat{K}_2} + \eta_1\theta_2\Big(\frac{\hat{K}_3 Q_n}{2\hat{K}_2} - \frac{Q_{n-1}}{2}\Big),\\
\zeta_{1,2} &= \theta_2\vartheta_1 + \theta_2\eta_{n-2},\\
\zeta_{1,3} &= 2(\theta_2\vartheta_1 + \theta_2\eta_{n-2}) + \sum_{k=1}^{2} C_{4-k}\vartheta_k + \theta_2\eta_{n-3},\\
\zeta_{1,j} &= \sum_{k=1}^{j-1}\theta_{j-k+1}\vartheta_k + \theta_2\eta_{n-j} + 2\sum_{k=1}^{j-2}\theta_{j-k}\vartheta_k + \theta_2\eta_{n-j-1} - \sum_{k=1}^{j-3}\theta_{j-k-1}\vartheta_k + \theta_2\eta_{n-j-2},\\
\zeta_{1,n-1} &= \eta_1\theta_2 + 2\Big(\sum_{k=1}^{n-3}\theta_{n-k-1}\vartheta_k + \theta_2\eta_2\Big) - \sum_{k=1}^{n-4}\theta_{n-k-2}\vartheta_k + \theta_2\eta_3,\\
\zeta_{1,n} &= -\frac{\hat{K}_1}{2\hat{K}_2} - \frac{\hat{K}_3\eta_1\theta_2}{\hat{K}_2} - \sum_{k=1}^{n-3}\theta_{n-k-1}\vartheta_k - \theta_2\eta_2,\\
\zeta_{2,2} &= \eta_{n-2},\\
\zeta_{2,3} &= \eta_{n-3} + 2\eta_{n-2},\\
\zeta_{2,j} &= \eta_{n-j} + 2\eta_{n-j+1} - \eta_{n-j+2},\ (4 \leqslant j \leqslant n-1),\\
\zeta_{3,3} &= 2(e_{n-3,n-1}\vartheta_1 + e_{n-3,2}\eta_{n-2}) + \sum_{k=1}^{2} e_{n-3,n-k}\vartheta_k + e_{n-3,2}\eta_{n-3},
\end{aligned}
$$

$$
\zeta_{i,j} = \sum_{k=1}^{j-1} e_{n-i,n-k}\vartheta_k + 2\Big(\sum_{k=1}^{j-2} e_{n-i,n-k}\vartheta_k\Big) + \sum_{k=1}^{j-3} e_{n-i,n-k}\vartheta_k + e_{n-i,2}(B_{n-j+2} + 2\eta_{n-j+1} + \eta_{n-j}),
$$
$$
(3 \leqslant i \leqslant [\tfrac{n+1}{2}];\ i \leqslant j \leqslant n+1-i;\ \textit{except}\ \zeta_{3,3}),
$$

in which

$$
\theta_i = \frac{\hat{K}_1\omega_i}{\hat{K}_2} - Q_i,\ (2 \leqslant i \leqslant n-1),\ w_i = -\frac{\hat{K}_3\omega_i}{\hat{K}_2} + \nu_i,\ (2 \leqslant i \leqslant n-1),
$$

$$
E_{i,j} = \begin{cases} -\dfrac{\hat{\Delta}_i\omega_j}{\hat{K}_2}, & i+j \neq n,\\[2mm] 1 - \dfrac{\hat{\Delta}_i\omega_j}{\hat{K}_2}, & i+j = n, \end{cases} \qquad (0 \leqslant i \leqslant n-3,\ 2 \leqslant j \leqslant n-1),
$$

$$
\vartheta_1 = 1/4,
$$

$$
\vartheta_j = (-1)^{j-1}4^{-j}\left(\sum_{t_1+2t_2+\cdots+(j-1)t_{j-1}=j-1}\frac{(t_1+\cdots+t_{j-1})!}{t_1!\cdots t_{j-1}!}(-4)^{(j-1)-t_1-\cdots-t_{j-1}}(4Q_2)^{t_1}\cdots(4Q_j)^{t_{j-1}}\right),
$$
$$
(2 \leqslant j \leqslant n-3),
$$

$$
\eta_1 = \frac{1}{U_2},
$$

$$
\eta_j = \frac{(-1)^{2j-1}\det\nabla_{n-1-j}([U_k]_{k=3}^{n-j+1}, 4, 4Q_2, \cdots, 4Q_{n-1-j})}{2^{n-1-j}U_2},\ (2 \leqslant j \leqslant n-2),
$$

$$
\det\nabla_1(U_3) = U_3,
$$

$$
\det\nabla_{n-1-j}([U_k]_{k=3}^{n-j+1}, 4, 4Q_2, \cdots, 4Q_{n-1-j}) = 2^{n-2-j}U_{n-j+1} + \sum_{p=1}^{n-2-j}(-1)^{2+p}2^{n-j-2-p}U_{n-j-p+1}
$$
$$
\cdot\left(\sum_{t_1+2t_2+\cdots+pt_p=p}\frac{(t_1+\cdots+t_p)!}{t_1!\cdots t_p!}(-4)^{p-t_1-\cdots-t_p}(4Q_2)^{t_1}\cdots(4Q_{p+1})^{t_p}\right),\ (2 \leqslant j \leqslant n-3),
$$

with $\omega_i(2 \leqslant i \leqslant n)$, $\nu_i(2 \leqslant i \leqslant n)$, $\hat{\Delta}_i(0 \leqslant i \leqslant n-3)$, $\hat{K}_1$, $\hat{K}_2$, $\hat{K}_3$ *as in Theorem 3.1,* $[x] = q$, $q \leqslant x < q+1$, q *is an integer.*

We can observe that $\mathbf{T}_{Q,n}^{-1}$ *is not only a symmetric matrix, but also a symmetric matrix along its secondary diagonal, i.e., persymmetric matrix.*

Proof. When $n > 5$, we can introduce the following two transformation matrices,

$$\hat{\mathcal{M}}_2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & -\frac{\hat{K}_3}{\hat{K}_2} & \ddots & \ddots & & & & \vdots \\ \vdots & 0 & 0 & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n},$$

$$\hat{\mathcal{N}}_2 = \begin{pmatrix} 1 & \frac{-\hat{K}_1}{2} & \theta_{n-1} & \theta_{n-2} & \cdots & \theta_3 & \theta_2 \\ 0 & 1 & -\frac{\omega_{n-1}}{\hat{K}_2} & -\frac{\omega_{n-2}}{\hat{K}_2} & \cdots & -\frac{\omega_3}{\hat{K}_2} & -\frac{\omega_2}{\hat{K}_2} \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n},$$

where

$$\theta_i = \frac{\hat{K}_1\omega_i}{2\hat{K}_2} - Q_i,\ (2 \leqslant i \leqslant n-1).$$

If we multiply $\hat{\mathcal{M}}_1\mathbf{T}_{Q,n}\hat{\mathcal{N}}_1$ by $\hat{\mathcal{M}}_2$ and $\hat{\mathcal{N}}_2$, the $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{N}}_1$ are as in the proof of Theorem 3.1, so we obtain

$$\hat{\mathcal{M}}_2\hat{\mathcal{M}}_1\mathbf{T}_{Q,n}\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2 = \begin{pmatrix} \frac{1}{2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \hat{K}_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & U_{n-1} & U_{n-2} & U_{n-3} & \cdots & U_5 & U_4 & U_3 & U_2 \\ \vdots & \vdots & 4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & 4Q_2 & 4 & \ddots & & & & & \vdots \\ \vdots & \vdots & 4Q_3 & 4Q_2 & \ddots & \ddots & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 4Q_{n-3} & \cdots & \cdots & \cdots & 4Q_3 & 4Q_2 & 4 & 0 \end{pmatrix}_{n\times n},$$

where

$$U_i = -\frac{\hat{K}_3\omega_i}{\hat{K}_2} + \nu_i,\ (2 \leqslant i \leqslant n-1),$$

we have

$$\hat{\mathcal{M}}_2\hat{\mathcal{M}}_1\mathbf{T}_{Q,n}\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2 = \digamma \oplus \urcorner,$$

where $\digamma = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \hat{K}_2 \end{pmatrix}_{2\times 2}$ is a diagonal matrix, and $\urcorner$ is a Toeplitz-like matrix

$$\urcorner = \begin{pmatrix} U_{n-1} & U_{n-2} & U_{n-3} & \cdots & U_5 & U_4 & U_3 & U_2 \\ 4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 4Q_2 & 4 & \ddots & & & & & \vdots \\ 4Q_3 & 4Q_2 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 4Q_{n-3} & \cdots & \cdots & \cdots & 4Q_3 & 4Q_2 & 4 & 0 \end{pmatrix}_{(n-2)\times(n-2)}.$$

$\digamma \oplus \urcorner$ is the direct sum of $\digamma$ and $\urcorner$. Let $\hat{\mathcal{M}} = \hat{\mathcal{M}}_2\hat{\mathcal{M}}_1$, $\hat{\mathcal{N}} = \hat{\mathcal{N}}_1\hat{\mathcal{N}}_2$, then we obtain

$$\mathbf{T}_{Q,n}^{-1} = \hat{\mathcal{N}}(\digamma^{-1} \oplus \urcorner^{-1})\hat{\mathcal{M}}, \tag{3.3}$$

$$\hat{\mathcal{M}} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{-Q_n}{2} & \vdots & & & & & & & \iddots & 1 \\ \frac{\hat{K}_3}{2\hat{K}_2}Q_n - \frac{Q_{n-1}}{2} & \vdots & & & & & & \iddots & 1 & -\frac{\hat{K}_3}{\hat{K}_2} \\ 0 & \vdots & & & & & \iddots & 1 & 2 & -1 \\ \vdots & \vdots & & & & \iddots & 1 & 2 & -1 & 0 \\ \vdots & \vdots & & & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & & & \vdots \\ \vdots & 0 & \iddots & \iddots & \iddots & \iddots & & & & \vdots \\ 0 & 1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}, \tag{3.4}$$

$$\hat{\mathcal{N}} = \begin{pmatrix} 1 & \frac{-\hat{K}_1}{2} & \theta_{n-1} & \theta_{n-2} & \cdots & \theta_3 & \theta_2 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \hat{\Delta}_{n-3} & e_{n-3,n-1} & e_{n-3,n-2} & \cdots & e_{n-3,3} & e_{n-3,2} \\ \vdots & \hat{\Delta}_{n-4} & e_{n-4,n-1} & e_{n-4,n-2} & \cdots & e_{n-4,3} & e_{n-4,2} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \hat{\Delta}_1 & e_{1,n-1} & e_{1,n-2} & \cdots & e_{1,3} & e_{1,2} \\ 0 & \hat{\Delta}_0 & e_{0,n-1} & e_{0,n-2} & \cdots & e_{0,3} & e_{0,2} \end{pmatrix}_{n\times n}, \tag{3.5}$$

where

$$e_{i,j}=\begin{cases} -\dfrac{\hat{\Delta}_i\omega_j}{\hat{K}_2}, & i+j\neq n,\\ 1-\dfrac{\hat{\Delta}_i\omega_j}{\hat{K}_2}, & i+j=n, \end{cases} \qquad (0\leqslant i\leqslant n-3,\ 2\leqslant j\leqslant n-1).$$

We can observe that the inverse matrix of $\digamma$ is $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \dfrac{1}{\hat{K}_2} \end{pmatrix}_{2\times 2}$.

According to Lemma 2.3 in [15], we have

$$\daleth^{-1}=\begin{pmatrix} 0 & \vartheta_1 & 0 & \cdots & \cdots & \cdots & 0\\ \vdots & \vartheta_2 & \vartheta_1 & \ddots & & & \vdots\\ \vdots & \vartheta_3 & \vartheta_2 & \ddots & \ddots & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & & \ddots & \vartheta_2 & \vartheta_1 & 0\\ 0 & \vartheta_{n-3} & \cdots & \cdots & \vartheta_3 & \vartheta_2 & \vartheta_1\\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{n-4} & \eta_{n-3} & \eta_{n-2} \end{pmatrix}_{(n-2)\times(n-2)},$$

where

$$\vartheta_1=1/4,$$

$$\vartheta_j=(-1)^{j-1}4^{-j}\left(\sum_{t_1+2t_2+\cdots+(j-1)t_{j-1}=j-1}\frac{(t_1+\cdots+t_{j-1})!}{t_1!\cdots t_{j-1}!}(-4)^{(j-1)-t_1-\cdots-t_{j-1}}(4Q_2)^{t_1}\cdots(4Q_j)^{t_{j-1}}\right),$$

$$(2\leqslant j\leqslant n-3),$$

$$\eta_1=\frac{1}{U_2},$$

$$\eta_j=\frac{(-1)^{2j-1}\det\nabla_{n-1-j}([U_k]_{k=3}^{n-j+1},4,4Q_2,\cdots,4Q_{n-1-j})}{2^{n-1-j}U_2},\ (2\leqslant j\leqslant n-2),$$

$$\det\nabla_1(U_3)=U_3,$$

$$\det\nabla_{n-1-j}([U_k]_{k=3}^{n-j+1},4,4Q_2,\cdots,4Q_{n-1-j})=2^{n-2-j}U_{n-j+1}+\sum_{p=1}^{n-2-j}(-1)^{2+p}2^{n-j-2-p}U_{n-j-p+1}$$

$$\cdot\left(\sum_{t_1+2t_2+\cdots+pt_p=p}\frac{(t_1+\cdots+t_p)!}{t_1!\cdots t_p!}(-4)^{p-t_1-\cdots-t_p}(4Q_2)^{t_1}\cdots(4Q_{p+1})^{t_p}\right),\ (2\leqslant j\leqslant n-3),$$

We obtain that

$$\digamma^{-1} \oplus \daleth^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{1}{\hat{K}_2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 0 & \vartheta_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vartheta_2 & \vartheta_1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vartheta_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & \vartheta_{n-3} & \cdots & \cdots & \vartheta_2 & \vartheta_1 \\ 0 & 0 & \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{n-3} & \eta_{n-2} \end{pmatrix}_{n\times n}. \tag{3.6}$$

According to formulas (3.3) (3.4) (3.5) (3.6) and the recurrence relations of the Pell-Lucas sequence, we obtain formula (3.2). Which completes the proof. □

4 Determinants and Inverses of the Persymmetric Pankel and Qankel Matrix

In this section, we give the determinants and inverses of the matrices $\mathbf{H}_{P,n}$ and $\mathbf{H}_{Q,n}$.

Theorem 4.1. *Let $\mathbf{H}_{P,n}$ be a persymmetric Pankel matrix as the form of* (1.3)*. Then we have*

$$\det \mathbf{H}_{P,1} = 1, \ \det \mathbf{H}_{P,2} = 3, \ \det \mathbf{H}_{P,3} = -8, \ \det \mathbf{H}_{P,4} = -20$$

and

$$\det \mathbf{H}_{P,n} = (-1)^{\frac{n^2+n-2}{2}} 2^{n-3}[K_2(P_{n-2} - P_{n-1}P_2) - K_3(P_{n-1} - P_nP_2)],$$

where K_2, K_3 are the same as in Theorem 1.1.

Proof. From (1.3), we have $\det \mathbf{H}_{P,n} = \det \mathbf{T}_{P,n} \det \hat{I}_n$. Then we can obtain this result by using Theorem 2.1 and $\det \hat{I}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Theorem 4.2. *Let $\mathbf{H}_{P,n}$ be an invertible persymmetric Pankel matrix and $n > 5$. Then we have*

$$\mathbf{H}_{P,n}^{-1} = \hat{I}_n \mathbf{T}_{P,n}^{-1}$$

where $\mathbf{T}_{P,n}^{-1}$ is the same as in Theorem 2.1.

Proof. We can obtain this conclusion by using (1.4) and Theorem 2.2. □

Theorem 4.3. *Let $\mathbf{H}_{Q,n}$ be a persymmetric Qankel matrix as the form of* (1.4)*. Then we have*

$$\det \mathbf{H}_{Q,1} = 2, \ \det \mathbf{H}_{Q,2} = -32, \ \det \mathbf{H}_{Q,3} = 480, \ \det \mathbf{H}_{Q,4} = -7186$$

and

$$\det \mathbf{T}_{Q,n} = 2(-1)^{n-1}4^{n-3}[\hat{K}_2(Q_{n-2} - Q_{n-1}Q_2) - \hat{K}_3(Q_{n-1} - Q_nQ_2)], \ (n > 5),$$

where $\hat{K}_2$, $\hat{K}_3$ are the same as in Theorem 3.1.

Proof. From (1.4), we have $\det \mathbf{H}_{Q,n} = \det \mathbf{T}_{Q,n} \det \hat{I}_n$. Then we can obtain this result by using Theorem 3.1 and $\det \hat{I}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Theorem 4.4. *Let $\mathbf{H}_{Q,n}$ be an invertible persymmetric Qankel matrix and $n > 5$. Then we have*

$$\mathbf{H}_{Q,n}^{-1} = \begin{pmatrix} \zeta_{1,n} & \zeta_{1,n-1} & \zeta_{1,n-2} & \cdots & \zeta_{1,3} & \zeta_{1,2} & \zeta_{1,1} \\ \zeta_{1,n-1} & \zeta_{2,n-1} & \zeta_{2,n-2} & \iddots & \zeta_{2,3} & \zeta_{2,2} & \zeta_{1,2} \\ \zeta_{1,n-2} & \zeta_{2,n-2} & \iddots & \iddots & \zeta_{3,3} & \zeta_{2,3} & \zeta_{1,3} \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \zeta_{1,3} & \zeta_{2,3} & \zeta_{3,3} & \iddots & \iddots & \zeta_{2,n-2} & \zeta_{1,n-2} \\ \zeta_{1,2} & \zeta_{2,2} & \zeta_{2,3} & \iddots & \zeta_{2,n-2} & \zeta_{2,n-1} & \zeta_{1,n-1} \\ \zeta_{1,1} & \zeta_{1,2} & \zeta_{1,3} & \cdots & \zeta_{1,n-2} & \zeta_{1,n-1} & \zeta_{1,n} \end{pmatrix}_{n\times n},$$

where $\zeta_{i,j}$ $(1 \leqslant i \leqslant \left[\frac{n+1}{2}\right];\ i \leqslant j \leqslant n+1-i)$ are the same as in Theorem 3.2.

Proof. We can obtain this conclusion by using (1.4) and Theorem 3.2. □

5 Numerical Example

In this section, two examples demonstrates the method which introduced above for the calculation of determinants and inverses of the symmetric Poeplitz and symmetric Qoeplitz matrix.

Example 1 We consider a 6×6 symmetric Poeplitz matrix:

$$\mathbf{T}_{P,6} = \begin{pmatrix} 1 & 2 & 5 & 12 & 29 & 70 \\ 2 & 1 & 2 & 5 & 12 & 29 \\ 5 & 2 & 1 & 2 & 5 & 12 \\ 12 & 5 & 2 & 1 & 2 & 5 \\ 29 & 12 & 5 & 2 & 1 & 2 \\ 70 & 29 & 12 & 5 & 2 & 1 \end{pmatrix}_{6\times 6}.$$

Using the corresponding formulas in Theorem 1, we get

$$K_2 = 907,\ K_3 = 376,\ y_2 = -46,\ x_2 = -111,$$

From (2.1), we obtain

$$\begin{aligned} \det \mathbf{T}_{P,6} &= (-1)^{6-1} 2^{6-3} [K_2(P_{6-2} - P_{6-1}P_2) - K_3(P_{6-1} - P_6 P_2)] \\ &= -112. \end{aligned}$$

As the inverse calculation, since 6 is even, from (2.3), we can get

$$\mathbf{T}_{P,6}^{-1} = \begin{pmatrix} -\frac{3}{7} & \frac{17}{14} & -\frac{4}{7} & \frac{3}{7} & -\frac{2}{7} & \frac{1}{14} \\ \frac{17}{14} & -\frac{27}{7} & \frac{39}{14} & -\frac{12}{7} & \frac{8}{7} & -\frac{2}{7} \\ -\frac{4}{7} & \frac{39}{14} & -\frac{31}{7} & \frac{43}{14} & -\frac{12}{7} & \frac{3}{7} \\ \frac{3}{7} & -\frac{12}{7} & \frac{43}{14} & -\frac{31}{7} & \frac{39}{14} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{8}{7} & -\frac{12}{7} & \frac{39}{14} & -\frac{27}{7} & \frac{17}{14} \\ \frac{1}{14} & -\frac{2}{7} & \frac{3}{7} & -\frac{4}{7} & \frac{17}{14} & -\frac{3}{7} \end{pmatrix}_{6\times 6}.$$

Example 2 We consider a 6×6 symmetric Qoeplitz matrix:

$$\mathbf{T}_{Q,6} = \begin{pmatrix} 2 & 6 & 14 & 34 & 82 & 198 \\ 6 & 2 & 6 & 14 & 34 & 82 \\ 14 & 6 & 2 & 6 & 14 & 34 \\ 34 & 14 & 6 & 2 & 6 & 14 \\ 82 & 34 & 14 & 6 & 2 & 6 \\ 198 & 82 & 34 & 14 & 6 & 2 \end{pmatrix}_{6\times 6}.$$

Using the corresponding formulas in Theorem 3.1, we get

$$\hat{K}_2 = 66192,\ \hat{K}_3 = 27432,\ \nu_2 = -212,\ \omega_2 = -512,$$

From (3.1), we obtain

$$\begin{aligned} \det \mathbf{T}_{Q,6} &= 2(-1)^{6-1}4^{6-3}[\hat{K}_2(Q_{6-2} - Q_{6-1}Q_2) - \hat{K}_3(Q_{6-1} - Q_6 Q_2)] \\ &= -1597440. \end{aligned}$$

As the inverse calculation, if we use the corresponding formulas in Theorem 3.2, we get

$$\zeta_{1,1} = -\tfrac{209}{3120},\ \zeta_{1,2} = \tfrac{79}{520},\ \zeta_{1,3} = \tfrac{41}{1560},\ \zeta_{1,4} = -\tfrac{11}{1560},\ \zeta_{1,5} = \tfrac{1}{520},\ \zeta_{1,6} = -\tfrac{1}{3120},$$

$$\zeta_{2,2} = -\tfrac{107}{260},\ \zeta_{2,3} = \tfrac{6}{65},\ \zeta_{2,4} = \tfrac{11}{260},\ \zeta_{2,5} = -\tfrac{3}{260},$$

$$\zeta_{3,3} = -\tfrac{329}{780},\ \zeta_{3,4} = \tfrac{37}{390}.$$

From (3.2), we can get

$$\mathbf{T}_{Q,6}^{-1} = \begin{pmatrix} -\frac{209}{3120} & \frac{79}{520} & \frac{41}{1560} & -\frac{11}{1560} & \frac{1}{520} & -\frac{1}{3120} \\ \frac{79}{520} & -\frac{107}{260} & \frac{6}{65} & \frac{11}{260} & -\frac{3}{260} & \frac{1}{520} \\ \frac{41}{1560} & \frac{6}{65} & -\frac{329}{780} & \frac{37}{390} & \frac{11}{260} & -\frac{11}{1560} \\ -\frac{11}{1560} & \frac{11}{260} & \frac{37}{390} & -\frac{329}{780} & \frac{6}{65} & \frac{41}{1560} \\ \frac{1}{520} & -\frac{3}{260} & \frac{11}{260} & \frac{6}{65} & -\frac{107}{260} & \frac{79}{520} \\ -\frac{1}{3120} & \frac{1}{520} & -\frac{11}{1560} & \frac{41}{1560} & \frac{79}{520} & -\frac{209}{3120} \end{pmatrix}_{6\times 6}.$$

Competing Interests

Author has declared that no competing interests exist.

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