British Journal of Mathematics & Computer Science
16(5): 1-23, 2016, Article no.BJMCS.24446
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

The Behavior of two Coupled Mechanical Oscillators in Non-linear Fields and the Possibility of Obtaining Controlled Halts

Nicuşor Nistor[1*], Constantin Gheorghieş[1], Nelu Cazacu[2] and Gabriela Tudor[3]

[1]*Department of Physics, Chemistry and Environment, Faculty of Science and Environment, "Dunărea de Jos" University of Galaţi, Romania.*
[2]*Department of Materials Science and Engineering, Faculty of Engineering, "Dunărea de Jos" University of Galaţi, Romania.*
[3]*"Calistrat Hogaş" National College of Tecuci, Galaţi, Romania.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/24446
Editor(s):
(1) Carlo Bianca, Laboratoire de Physique Théorique de la Matière Condensée, Sorbonne Universités, France.
(2) Paul Bracken, Department of Mathematics, The University of Texas-Pan American, Edinburg, TX 78539, USA.
Reviewers:
(1) Zarqa Bano, Sukkur IBA, Sindh, Pakistan.
(2) Jessé Rebello de Souza Junior, University of São Paulo, Brazil.
(3) Anonymous, Motilal Nehru National Institute of Technology, India.
(4) Anonymous, Aristotle University, Greece.
(5) Alal Hosen, Rajshahi University of Engineering and Technology, Bangladesh.
Complete Peer review History: http://sciencedomain.org/review-history/14572

Original Research Article

Received: 20th January 2016
Accepted: 23rd April 2016
Published: 11th May 2016

Abstract

The proposed mechanical system consists of two magnetic oscillating subsystems, which are mechanically coupled. The first one consists of a spring-magnet-mass subsystem and the other one is a spring-mass subsystem. The non-linear symmetric field created by two other fixed magnets, oriented for attraction, acts only upon the first subsystem. The entire system can oscillate horizontally, without friction and without loss of energy. Oscillations occur with conservation of kinetic energy and potential energy stored in the springs. During the movement, depending on the amplitude of oscillations, the main body stops for a period and this can be controlled by the physical constants of the system. The reason for these controlled halts of oscillations is the transfer of mechanical impulse, due to both oscillators entering

*Corresponding author: E-mail: Nicusor.Nistor@ugal.ro;

in the frequencies equalizing status with different initial frequencies. The frequency of the main oscillator gets synchronized with the frequency of the second oscillator, by its deeper or more superficial presence in a non-linear magnetic field, which is strongly dependent on distance. This paper will apply linear algebra transformation methods, on the R^3 vector spaces, for differential equations systems, that are applicable to non-linear systems only if we consider some special conditions. The general case will be solved and the method will be verified through a numerical application.

Keywords: Nonlinear second order differential equation system; elliptic functions; 2N parametric vector space, R^3 vector spaces; double oscillating mechanical system; elastic forces field; magnetic forces field; controlled halts.

1 Introduction

The proposed mechanical system consists of two oscillating subsystems, which are mechanically coupled, like in the Fig. 1. The first subsystem will be main one, consisting of one spring and a mass with a magnet inside, while the second one is a classical spring-mass subsystem. Both subsystems interact with each other through a connection spring.

The oscillations that occur in various situations are very interesting and will be described in the following. We will treat the ideal case, with no friction and in the assumption that there is no loss of energy during the magnetic and mechanic interactions. One of the most important characteristic is that the system starts with an applied initial impulse and kinetic energy is transformed periodically into potential energy.

1.1 The description of the mechanical system

We will start from the figure below, which displays the proposed mechanical system, where there are two mechanical subsystems being in nonlinear interaction, as further shown:

- The first subsystem, called the main subsystem, consists of mass-body m_1 and a spring that will introduce a linear elastic field. At the same time, the subsystem is placed in a magnetic field, nonlinear strong at the ends of the race, thus creating a nonlinear potential field. In the magnetic force description, a simple monopole-monopole expression has been used, for long bar magnets, described in [1].
- The second subsystem, called secondary subsystem, consists of mass-body m_2 and a resort that will produce oscillations under the influence of the potential field of its own resort, but also under the action of the inertial field induced by the first oscillating system.

In the mathematical description of the introduced mechanical system we are taking into account some equations from the physical model already described in the authors' earlier work [2]. Actually, the current study represents a continuation of ideas presented in that article, at a more complex level.

We have to specify that, during the oscillations, the magnetic field acts only on the ensemble formed by m_1 mass and its interior magnet. The permanent magnet which is inside the m_1 body, is oriented as shown and the entire mechanical system can move freely, without friction, in X axis direction.

1.2 The constants of this mechanical system are

k_1 – is the elastic constant of the first spring;

k_2 – is the elastic constant of the second spring;

k_3 – is the magnetic constant depending on strength of the permanent magnet;

m_1 – is mass of the magnet-mass body of the principal oscillator;

m_2 – is mass of the mass-body of the second oscillator;

ε - is a magnetic constant depending on the mechanical construction of the system.

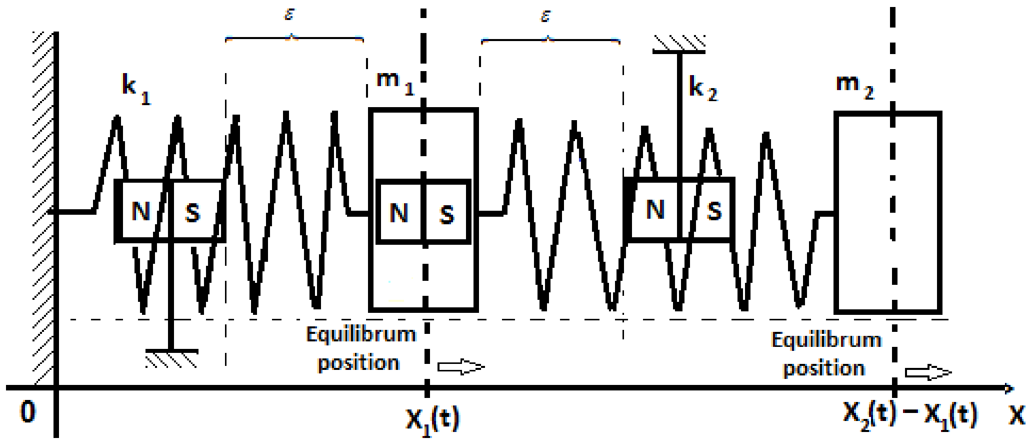

Fig. 1. Two coupled spring-mass subsystems, interacting in a nonlinear magnetic field

From a mechanical point of view, the mathematical differential equations that describe the system include the resultant of the static and dynamic forces, as in the following, and some constructive ideas have been taken from scientific papers [3] and [4].

1.3 The involved forces are

F_{e1} – the elastic force of the first subsystem, given by the expression:

$$F_{e1} = -k_1 X_1(t) + k_2[X_2(t) - X_1(t)] \tag{1}$$

F_{e2} – the elastic force of the second subsystem, with the expression:

$$F_{e2} = -k_2[X_2(t) - X_1(t)] \tag{2}$$

In the way of writing the equations that describe the elastic field created by springs, we took into account for the lineat part of our problem, the work [5].

Fm_n – the magnetic force acting on the first permanent magnet 1 with nonlinear expression, it has been also described in detail, in the work [2], and has the expression:

$$Fm_n(X_1) = \frac{k_3}{[\varepsilon - X_1(t)]^2} - \frac{k_3}{[\varepsilon + X_1(t)]^2} \tag{3}$$

$F_{r1} = Fe_1 + Fm_n$ – the static forces resultant of the first subsystem:

$$F_{r1}=-k_1X_1(t)+k_2[X_2(t)-X_1(t)]+\frac{k_3}{[\varepsilon-X_1(t)]^2}-\frac{k_3}{[\varepsilon+X_1(t)]^2} \tag{4}$$

$F_{r_2}=Fe_2$ – the static forces resultant of the second subsystem:

$$F_{r2}=-k_2[X_2(t)-X_1(t)] \tag{5}$$

F_{i_1} – the inertial force acting on the first mass:

$$F_{i_1}=m_1\frac{d^2X_1(t)}{dt^2} \tag{6}$$

F_{i_2} – the inertial force acting on the second mass:

$$F_{i_2}=m_2\frac{d^2X_2(t)}{dt^2} \tag{7}$$

2 Applied Methods

Given the forces described in the expressions above, we can say that the system of differential equations describing the mechanical system has the following structure:

$$\begin{cases} Fi_1=Fe_1+Fm_n \\ Fi_2=Fe_2 \end{cases} \Leftrightarrow \begin{cases} m_1\dfrac{d^2X_1(t)}{dt^2}=-k_1X_1(t)+k_2[X_2(t)-X_1(t)]+\dfrac{k_3}{[\varepsilon-X_1(t)]^2}-\dfrac{k_3}{[\varepsilon+X_1(t)]^2} \\ m_2\dfrac{d^2X_2(t)}{dt^2}=-k_2[X_2(t)-X_1(t)] \end{cases} \tag{8}$$

We have an homogenous, non-linear, of 2nd-order differential system, with coupled equations, difficult to solve through an analytical method. In order to solve these equations, the model proposed by the authors can be described in the following way:

We will rewrite the system in the form of a matrix system as follows:

$$\begin{pmatrix} m_1\dfrac{d^2X_1(t)}{dt^2} \\ m_2\dfrac{d^2X_1(t)}{dt^2} \end{pmatrix}=\begin{pmatrix} -(k_1+k_2) & k_3 & k_2 \\ k_2 & 0 & -k_2 \end{pmatrix}\begin{pmatrix} X_1(t) \\ \dfrac{1}{[\varepsilon-X_1(t)]^2}-\dfrac{1}{[\varepsilon+X_1(t)]^2} \\ X_2(t) \end{pmatrix} \tag{9}$$

As the system is written in a 2x3 matrix form, with 2 rows and 3 columns and we want to make a 2N–parametric vector transformation, in the vector space R^3, we will perform a mathematical artifice. We consider a nonlinear mathematical function: $Fm_n(X_1) = \frac{d^2 G_n(X_1)}{dt^2}$, with the following form, described also in (3):

$$\frac{d^2 G_n(X_1)}{dt^2} = \frac{k_3}{[\varepsilon - X_1(t)]^2} - \frac{k_3}{[\varepsilon + X_1(t)]^2} \tag{10}$$

The system becomes:

$$\begin{pmatrix} m_1 \frac{d^2 X_1(t)}{dt^2} \\ \frac{d^2 G_n(X_1)}{dt^2} \\ m_2 \frac{d^2 X_1(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} -(k_1+k_2) & k_3 & k_2 \\ 0 & k_3 & 0 \\ k_2 & 0 & -k_2 \end{pmatrix} \begin{pmatrix} X_1(t) \\ \frac{1}{[\varepsilon - X_1(t)]^2} - \frac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \tag{11}$$

We divide the conexion matrix by k_2, as follows:

$$\begin{pmatrix} m_1 \frac{d^2 X_1(t)}{dt^2} \\ \frac{d^2 G_n(X_1)}{dt^2} \\ m_2 \frac{d^2 X_2(t)}{dt^2} \end{pmatrix} = k_2 \begin{pmatrix} -\frac{(k_1+k_2)}{k_2} & \frac{k_3}{k_2} & \frac{k_2}{k_2} \\ 0 & \frac{k_3}{k_2} & 0 \\ \frac{k_2}{k_2} & 0 & -\frac{k_2}{k_2} \end{pmatrix} \begin{pmatrix} X_1(t) \\ \frac{1}{[\varepsilon - X_1(t)]^2} - \frac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \tag{12}$$

Denoting with $q = \frac{k_1}{k_2}$ and $n = \frac{k_3}{k_2}$, we obtain the next system written in E - orthogonal vector basis:

$$E: \begin{pmatrix} \frac{m_1}{k_2} \frac{d^2 X_1(t)}{dt^2} \\ \frac{1}{k_2} \frac{d^2 G_n(X_1)}{dt^2} \\ \frac{m_2}{k_2} \frac{d^2 X_2(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} -(q+1) & n & 1 \\ 0 & n & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1(t) \\ \frac{1}{[\varepsilon - X_1(t)]^2} - \frac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \tag{13}$$

We are going to transform the system of differential equations in a quasi-diagonal form. This fact will help us to solve those three differential equations separately, in a special vector space, which will be found further down.

This method, which we are going to apply, was already successfully applied in the earlier authors' work [6], where a simple 1N parametric vector transformation in R^2 vector space, was considered.

The problem that implies the usage of vector spaces in solving of systems of liner differential equations, was widely described in specialty scientific paper and also in an original manner in work [7].

We are seeking to transform matrix A, into a matrix B, following a vector transformation, using the associated matrix of a V-vector space together with his inverse, V^{-1}, while observing that the following condition must be fulfilled:

$$\det(A) = \det(B) \tag{14}$$

We are searching to transform the matrix A,

$$A = \begin{pmatrix} -(q+1) & n & 1 \\ 0 & n & 0 \\ 1 & 0 & -1 \end{pmatrix} \tag{15}$$

In a special, quasi-diagonal form one:

$$B = \begin{pmatrix} \frac{\sqrt{q^2+4}-q-2}{2} & 0 & 0 \\ -q & n & 0 \\ 0 & 0 & \frac{-\sqrt{q^2+4}-q-2}{2} \end{pmatrix} \tag{16}$$

In connection with B matrix, we specify that:

- We have chosen this form of quasi – diagonal, B- matrix, because we intend to solve this problem (a 2nd-order, nonlinear differential equations system), by solved model presented in [6], and we want to bring the system of differential nonlinear equations, to almost that form, with specification that, here we use a quasi-diagonal form one, not a diagonal form.
- The matrix B, described in equation (16), was obtained after several successive attempts, not applying a certain known algorithm, just based on the matrix multiplications, conveniently chosen. It was obtained by thinking in advance the steps for solving mathematical model.

This solution allows us to symbolically decouple those three differential equations, and in compliance with the conditions stated above and as it is written below, where the resemblance relations of matrix A and B are being verified:

$$\begin{cases} \det(A) = nq \\ \det(B) = \frac{n\left(\sqrt{q^2+4}-q-2\right)\left(-\sqrt{q^2+4}-q-2\right)}{4} = \frac{n\left[-q^2-4-\left(q^2+4q+4\right)\right]}{4} = nq \end{cases} \tag{17}$$

Looking for a three-dimensional V-basis, in R^3 vector space, the: (a,b,c,d,e,f,g,h,i) - independent parameters, will be found as follows:

- We are searching to find a generalized vector transformation, regardless of the physical constants of the system, a matrix form, with nine unknown matrix characteristic in R^3 -vector space.
- In order to perform an advance unitary transformation, parametrically point of view, we will consider all the numerical possibilities, through the nine parameters taken into account, and this it was the reason of choosing that those nine parameters.

$$V = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \tag{18}$$

Calculating symbolically the inverse of this matrix, we find the following form:

$$V^{-1} = \begin{pmatrix} \frac{ei-fh}{aei-afh-dbi+dch+gbf-gce} & -\frac{bi-ch}{aei-afh-dbi+dch+gbf-gce} & -\frac{bf+ce}{aei-afh-dbi+dch+gbf-gce} \\ -\frac{di-fg}{aei-afh-dbi+dch+gbf-gce} & \frac{ai-cg}{aei-afh-dbi+dch+gbf-gce} & -\frac{af-cd}{aei-afh-dbi+dch+gbf-gce} \\ -\frac{-dh+eg}{aei-afh-dbi+dch+gbf-gce} & -\frac{ah-bg}{aei-afh-dbi+dch+gbf-gce} & \frac{ae-bd}{aei-afh-dbi+dch+gbf-gce} \end{pmatrix} \tag{19}$$

where for any R^3 - vector space form, determined by physical constants variations, verifies the equality:

$$V^{-1}V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{20}$$

According to the theory of vector space transformation, used in linear algebra, we obtain:

$$V \ A \ V^{-1} = B \tag{21}$$

- The intermediate calculus, regarding the matrix multiplication vector rule, is too voluminous, and we chose not to mention there, because we shouldnt complicated the calculus, but each elements of this matrix (eq_1 ... eq_2), will be described in (23). .. (31), and we specify that, its verification is immediate.
- We now can write the matrix multiplication, as following:

$$V \ A \ V^{-1} = \begin{pmatrix} eq_{11} & eq_{12} & eq_{13} \\ eq_{21} & eq_{22} & eq_{23} \\ eq_{31} & eq_{32} & eq_{33} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{q^2+4}-q-2}{2} & 0 & 0 \\ -q & n & 0 \\ 0 & 0 & \frac{-\sqrt{q^2+4}-q-2}{2} \end{pmatrix} \tag{22}$$

Based on the formula above which transforms a matrix according to a vector space and its inverse, we obtain after successive multiplications, next nine equations with nine unknowns:

$$eq_{11}: \left(\frac{-aeiq-aei+afhq+afh+baf-cae+nedi-nfdh-ndbi+ndch+ieg-hfg-gbf+gce}{aei-afh-dbi+dch+gbf-gce} = \frac{\sqrt{q^2+4}-q-2}{2} \right) \tag{23}$$

$$eq_{12}:\left(\frac{-beiq-bei+bfhq+b^2f-bce+ne^2i-nefh-nebi+nech+hei-fh^2+hce}{aei-afh-dbi+dch+gbf-gce}=0\right) \tag{24}$$

$$eq_{13}:\left(-\frac{ceiq-cfhq-cfh-cbf+c^2e-nfei+nf^2h+nfbi-nfch-ei^2+ifh}{aei-afh-dbi+dch+gbf-gce}=0\right) \tag{25}$$

$$eq_{21}:\left(\frac{idqa+adi-afgq-a^2f+acd-nd^2i+ndfg+ndai-ndcg-gdi+fg^2-\mathrm{gcd}}{aei-afh-dbi+dch+gbf-gce}=-q\right) \tag{26}$$

$$eq_{22}:\left(\frac{dbiq+dbi-bfgq+gbf+baf-cbd-nedi+nfeg+neai-necg+idh-gfg-afh-dch}{aei-afh-dbi+dch+gbf-gce}=n\right) \tag{27}$$

$$eq_{23}:\left(\frac{cidq-cfgq-cfg-caf+c^2d-nfdi+nf^2g+nfai-nfcg-di^2+ifg+iaf}{aei-afh-dbi+dch+gbf-gce}=0\right) \tag{28}$$

$$eq_{31}:\left(\frac{-adhq-adh+aegq+a^2e-abd+nd^2h-dneg-ndah+ndbg+gdh-eg^2+gdb}{aei-afh-dbi+dch+gbf-gce}=0\right) \tag{29}$$

$$eq_{32}:\left(\frac{-bdhq+begq+beg+bae-b^2d+nedh-ne^2g-neah+nebg+dh^2-heg-hae}{aei-afh-dbi+dch+gbf-gce}=0\right) \tag{30}$$

$$eq_{33}:\left(\frac{-cdhq-dch+cegq+gce+cae-cbd+nfdh-nfeg-nfah+nfbg+idh-ieg-aei+dbi}{aei-afh-dbi+dch+gbf-gce}=\frac{-\sqrt{q^2+4}-q-2}{2}\right) \tag{31}$$

We form the next system:

$$S=\{eq_{11},eq_{12},eq_{13},eq_{21},eq_{22},eq_{23},eq_{31},eq_{32},eq_{33}\} \tag{32}$$

Were we have to mention that solving calculus of (eq_1 ... eq_2), is also too voluminous, and we chosed not to mention there, because is an elementary one. After solving the nine equations system, with nine unknowns, depending on two parameters (q, n), we obtain solutions for those 9 unknowns as follows:

$$\text{Solutions}:\left\{\begin{array}{ccc} a=-q & b=n+1 & c=\dfrac{q+\sqrt{q^2+4}}{2} \\ d=\dfrac{2n+2+q+q\sqrt{q^2+4}}{2} & e=\dfrac{n^2+2n+nq+q}{2} & f=0 \\ g=0 & h=1 & i=-1 \end{array}\right\} \tag{33}$$

It follows that the three-dimensional matrix, associated with V- vector space, which helped us to perform the transformation is V:

$$V = \begin{pmatrix} -q & n+1 & \dfrac{q+\sqrt{q^2+4}}{2} \\ \dfrac{2n+2+q+q\sqrt{q^2+4}}{2} & \dfrac{n^2+2n+nq+q}{2} & 0 \\ 0 & 1 & -1 \end{pmatrix} \tag{34}$$

Discution:

- The associated matrix of V-vector space thus found has a parametric form, depending on initially given constants, and it can turn the connections matrix A, into a connections matrix B, in any situation, regardless of the values of the physical constants (K_1, K_2 , K_3), with only one condition that the V–matrix be non singular.
- V and his inverse V^{-1}, are parametric type and by reason, it is possible to transform matrix A into matrix B, in the general case, regardless of the physical constants of the system.

With matrix B, this 2[nd] order differential system, described in form of equation (13), becomes:

$$V: \begin{pmatrix} \dfrac{m_1}{k_2}\dfrac{d^2X_1(t)}{dt^2} \\ \dfrac{1}{k_2}\dfrac{d^2G_n(X_1)}{dt^2} \\ \dfrac{m_2}{k_2}\dfrac{d^2X_2(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{q^2+4}-q-2}{2} & 0 & 0 \\ -q & n & 0 \\ 0 & 0 & \dfrac{-\sqrt{q^2+4}-q-2}{2} \end{pmatrix} \begin{pmatrix} X_1(t) \\ \dfrac{1}{[\varepsilon - X_1(t)]^2} - \dfrac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \tag{35}$$

Which involves solving a system of a 3 - differential equations, with this decoupled form:

$$\begin{cases} a)\ \dfrac{m_1}{k_2}\dfrac{d^2X_1(t)}{dt^2} = \dfrac{\sqrt{q^2+4}-q-2}{2}X_1(t) \\ b)\ \dfrac{1}{k_2}\dfrac{d^2G_n(X_1)}{dt^2} = -qX_1(t) + \dfrac{n}{[\varepsilon - X_1(t)]^2} - \dfrac{n}{[\varepsilon + X_1(t)]^2} \\ c)\ \dfrac{m_2}{k_2}\dfrac{d^2X_2(t)}{dt^2} = \dfrac{-\sqrt{q^2+4}-q-2}{2}X_2(t) \end{cases} \tag{36}$$

These three differential equations will be separately solved, as follows:

a) The first 2[nd] order linear equation, written in the following form:

$$\frac{d^2X_1(t)}{dt^2} - \frac{k_2}{m_1}\cdot\frac{\sqrt{q^2+4}-q-2}{2}X_1(t) = 0 \tag{37}$$

Where we denote the first oscillator's own pulsation:

$$\omega_1^2 = -\frac{k_2}{m_1} \cdot \frac{\sqrt{q^2+4}-q-2}{2} \tag{38}$$

The general solution of differential equation is:

$$X_1(t) = \zeta_1 e^{I\omega_1 t} + \zeta_2 e^{-I\omega_1 t} \tag{39}$$

Where, after transformation, using Euler's formulas, we arrive at:

$$X_1(t) = x_{01} \sin(\omega_1 t + \varphi_1) \tag{40}$$

b) The second nonlinear differential equation, in this uncoupled form, will have the following resolution, based on the authors' earlier work [2], where the motion of a single spring-mass-magnet in magnetic field was considered, and where analytical solution of elliptic sine form functions was found.

In the description of the Jacobi elliptical functions, involved by this problem resolution, what has been taken into account is the dedicated content of two scientific papers [8] and [9].

With the purpose of finding the form for the analytic solution of differential equation, we have to proceed as shown below. Equation solving metod presented in [2], will be summarized in the following way:

$$\frac{d^2 G_n(X_1)}{dt^2} = -qk_2 X_1(t) + \frac{k_2 n}{[\varepsilon - X_1(t)]^2} - \frac{k_2 n}{[\varepsilon + X_1(t)]^2} = -k_1 X_1(t) + \frac{k_3}{[\varepsilon - X_1(t)]^2} - \frac{k_3}{[\varepsilon + X_1(t)]^2} \tag{41}$$

We firstly calculate the x_{1p} and p parameters from the roots of the right member of upper differential equation:

$$x_{1p} = Root\left\{ -k_1 X_1 + \frac{k_3}{[\varepsilon - X_1]^2} - \frac{k_3}{[\varepsilon + X_1]^2} = 0 \right\} \tag{42}$$

and

$$p = Root\left[-(1+p^2k^2)x_{1p} + 2p^2k^2x_{1p}^3 = 0 \right] \tag{43}$$

After this we get the eccentricity argument of elliptic function as follows:

$$k(p, x_0) = \frac{1}{p\sqrt{2x_0^2 - 1}} \tag{44}$$

We can calculate then the time argument of elliptic function in this mode:

If we consider, the initial gived function, wich is the right member of (41), as F_r :

$$F_r = -k_1 X_1 + \frac{k_3}{[\varepsilon - X_1]^2} - \frac{k_3}{[\varepsilon + X_1]^2} \tag{45}$$

We find a special prototype function - F_p, starting from our dedicated form of the right member as follows:

$$F_p(u, v, p, k, x_0) = u \left[-(1 + p^2 k^2) x_0 \right] + v \left(2 p^2 k^2 x_0^3 \right) \tag{46}$$

For finding (u, v) - constants, the next system of integral equations is implied, in order to solve our problem:

$$\begin{cases} F_p = F_r \Leftrightarrow -u(1+p^2k^2)x_p + 2vp^2k^2x_p^3 = k_1 x_{1p} + \dfrac{k_3}{[\varepsilon - x_{1p}]^2} - \dfrac{k_3}{[\varepsilon + x_{1p}]^2} \\ \displaystyle\int_0^{x_p} F_p \, dX = \int_0^{x_p} F_r \, dX \end{cases} \tag{47}$$

With specification that the first equation is solved for: x_p, p, k - being constants, and the second one is integrated for variable X, and for constants: p, k.

After the above system is solved for finding (u, v) - constants, we have to identify the coefficients of the expression, provided from the right member of (41):

$$-x_0\alpha^2 X + 2x_0\alpha^2 k^2 X^3 - x_0\alpha^2 k^2 X = -u(1 + p^2 k^3)X + 2vp^2k^2X^3 \tag{48}$$

Starting from the system:

$$\begin{cases} -x_0\alpha^2 - x_0\alpha^2 k^2 = -u(1 + p^2 k^3) \\ 2x_0\alpha^2 k^2 = 2vp^2k^2 \end{cases} \tag{49}$$

where after solving it for α, we will be able to obtain the final form of the function.

$$G_n(X_1(t)) = x_{01} \; sn \left[\frac{\zeta_3}{\sqrt{x_{01}}} \; t, \; \frac{\zeta_4}{\sqrt{2x_{01}^2 - 1}} \right] \tag{50}$$

Wich is a Jacobi sine, elliptic solution of our partial problema.

In the previously formulated problem, what has been taken into account is a number of some constructive ideas from the authors' research work [10,11]. The problem resolution consists of finding an analitical general formula to get the interaction between linear and nonlinear subystems.

c) The third one is a second order linear equation, and is written in the form:

$$\frac{d^2X_2(t)}{dt^2}-\frac{k_2}{m_2}\cdot\frac{\left(-\sqrt{q^2+4}-q-2\right)}{2}X_2(t)=0 \tag{51}$$

We denote the second oscillator's own pulsation:

$$\omega_2^2=-\frac{k_2}{m_2}\cdot\frac{\left(-\sqrt{q^2+4}-q-2\right)}{2} \tag{52}$$

The general solution of this differential equation is:

$$X_2(t)=\zeta_5 e^{I\omega_2 t}+\zeta_6 e^{-I\omega_2 t} \tag{53}$$

After transformation, using Euler formulas, we get:

$$X_2(t)=x_{02}\sin(\omega_2 t+\varphi_2) \tag{54}$$

as we obtain the following system sollutions, written in V basis:

$$V:\begin{cases} X_1(t)=x_{01}\sin(\omega_1 t+\varphi_1) \\ G_n(X_1(t))=x_{01}\ sn\left[\frac{\zeta_3}{\sqrt{x_{01}}}\ t,\ \frac{\zeta_4}{\sqrt{2x_{01}^2-1}}\right] \\ X_2(t)=x_{02}\sin(\omega_2 t+\varphi_2) \end{cases} \tag{55}$$

Finding the final form of those two solutions involves a basic active transformation:

$$E:\underbrace{\begin{pmatrix} X_1(t) \\ G_n(X_1) \\ X_2(t) \end{pmatrix}}_{\substack{\text{Searched} \\ \text{solutions}}}=\underbrace{\begin{pmatrix} -q & n+1 & \frac{q+\sqrt{q^2+4}}{2} \\ \frac{2n+2+q+q\sqrt{q^2+4}}{2} & \frac{n^2+2n+nq+q}{2} & 0 \\ 0 & 1 & -1 \end{pmatrix}}_{\text{V matrix, associated to } \vec{\text{V}} \text{ - vector spaces}}\cdot\underbrace{\begin{pmatrix} x_{01}\sin(\omega_1 t+\varphi_1) \\ x_{01}\ sn\left[\frac{\zeta_3}{\sqrt{x_{01}}}\ t,\ \frac{\zeta_4}{\sqrt{2x_{01}^2-1}}\right] \\ x_{02}\sin(\omega_2 t+\varphi_2) \end{pmatrix}}_{\text{Decuplated form of solutions}} \tag{56}$$

Discusion:

- We use the inverse vector transformation method, of differential equations, which has been described also in other research work, where we can mention work [7].
- We will return to the basic vector space for finding the general solution to this problem in the initial given conditions.
- After inverse transformation, the unknown function: $G_n(X_1)$, will have no effect and will not be written.

We finally find the general solution in E space, as follows

$$E:\begin{cases} X_1(t) = -q\ x_{01}\sin(\omega_1 t+\varphi_1)+(n+1)x_{01}\ sn\left[\frac{\zeta_3}{\sqrt{x_{01}}}\ t,\ \frac{\zeta_4}{\sqrt{2x_{01}^2-1}}\right]+\frac{q+\sqrt{q^2+4}}{2}x_{02}\sin(\omega_2 t+\varphi_2) \\ X_2(t) = x_{01}\ sn\left[\frac{\zeta_3}{\sqrt{x_{01}}}\ t,\ \frac{\zeta_4}{\sqrt{2x_{01}^2-1}}\right]-x_{02}\sin(\omega_2 t+\varphi_2) \end{cases} \tag{57}$$

3 A Specific Numerical Application

With the purpose of applying into practice the analytical solution presented above, we take a numerical application, so that the double nonlinear oscillator has the described behavior.

We start to give numerical values of physical constants, to eliminate the parameters considered in the beginning, and to highlight the existence of the functional halts.

Although there are many other possible physical values of the constants, we have selected the next set of values, because the obtained effect is similar to the practical experimental situation, from which we have started, and we developed the theoretical study.

$$Numerical\ \ Values = \begin{cases} \varepsilon = 5mm \\ m_1 = 1.265g \\ m_2 = 11.81g \\ k_1 = 0.585mN/mm \\ k_2 = 1mN/mm \\ k_3 = 1mN/mm \end{cases} \tag{58}$$

We denote by:

$$\mathrm{q} = \frac{\mathrm{k}_1}{\mathrm{k}_2} = 0.585 \tag{59}$$

and:

$$\mathrm{n} = \frac{\mathrm{k}_3}{\mathrm{k}_2} = 1 \tag{60}$$

The system of nonlinear second order differential equations becomes:

$$\begin{cases} 1.265\,\frac{d^2X_1(t)}{dt^2} = -0.585\,X_1(t)+[X_2(t)-X_1(t)]+\frac{1}{[5-X_1(t)]^2}-\frac{1}{[5+X_1(t)]^2} \\ \frac{\mathrm{d}^2 G_{\mathrm{n}}(\mathrm{X}_1)}{\mathrm{dt}^2} = \frac{1}{[5-X_1(t)]^2}-\frac{1}{[5+X_1(t)]^2} \\ 11.81\,\frac{d^2X_2(t)}{dt^2} = -[X_2(t)-X_1(t)] \end{cases} \tag{61}$$

Matrix B will take the form:

$$B = \begin{pmatrix} -0.25 & 0 & 0 \\ -0.585 & 1 & 0 \\ 0 & 0 & -2.334 \end{pmatrix} \tag{62}$$

The vector space that decouples the solutions of our system will be:

$$V = \begin{pmatrix} -0.585 & 2 & 1.334 \\ 2.9 & 2.085 & 0 \\ 0 & 1 & -1 \end{pmatrix} \tag{63}$$

Our system is rewritten in a quasi-diagonal form, in the vector space thus founded:

$$\begin{pmatrix} 1.265\dfrac{d^2 X_1(t)}{dt^2} \\ \dfrac{d^2 G_n(X_1)}{dt^2} \\ 11.8\dfrac{d^2 X_2(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} -0.25 & 0 & 0 \\ -0.585 & 1 & 0 \\ 0 & 0 & -2.334 \end{pmatrix} \begin{pmatrix} X_1(t) \\ \dfrac{1}{[5-X_1(t)]^2} - \dfrac{1}{[5+X_1(t)]^2} \\ X_2(t) \end{pmatrix} \tag{64}$$

This implies solving three isolated differential equations as follows:

$$\begin{cases} a)\ 1.265\dfrac{d^2 X_1(t)}{dt^2} = -0.25 \quad X_1(t) \\ b)\ \dfrac{d^2 G_n(X_1)}{dt^2} = -0.585 \quad X_1(t) + \dfrac{1}{[5-X_1(t)]^2} - \dfrac{1}{[5+X_1(t)]^2} \\ c)\ 11.8\dfrac{d^2 X_2(t)}{dt^2} = -2.334 \quad X_2(t) \end{cases} \tag{65}$$

We solve separately these tree differential equations:

a) The first 2nd order linear differential equation, having the proper pulsation:

$$\omega_1^2 = \frac{0.25}{1.256} \Leftrightarrow \omega_1 = \sqrt{\frac{0.25}{1.256}} = 0.444 \ \text{rad/s} \tag{66}$$

With solutions:

$$X_1(t) = x_{01}\sin(0.444 \quad t) \tag{67}$$

b) The second order, non-linear differential equation which have elliptical solution, are solved in the following:

Ecuation (42) and (43) are becaming:

$$x_p = Root(F_r = 0) = (-5.554, -4.376, \ 0, 4.376, 5.554) \tag{68}$$

And:

$$p = Root\left[-(1+p^2)(4.376) + 2p^2(4.376)^3 = 0\right] = 0.1637 \tag{69}$$

For the eccentricity argument we get:

$$k(p, x_0) = \frac{6.107}{\sqrt{2x_0^2 - 1}} \tag{70}$$

The (45) and (46) form becomes:

$$F_r(X_1) = -0.585 \ \ X_1(t) + \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \tag{71}$$

The found prototype function will be:

$$F_p(X_1) = -0.89635789 X_1 + 0.04679999 X_1^3 \tag{72}$$

The both functions have the same roots, and comparison graph of the two functions is shown below.

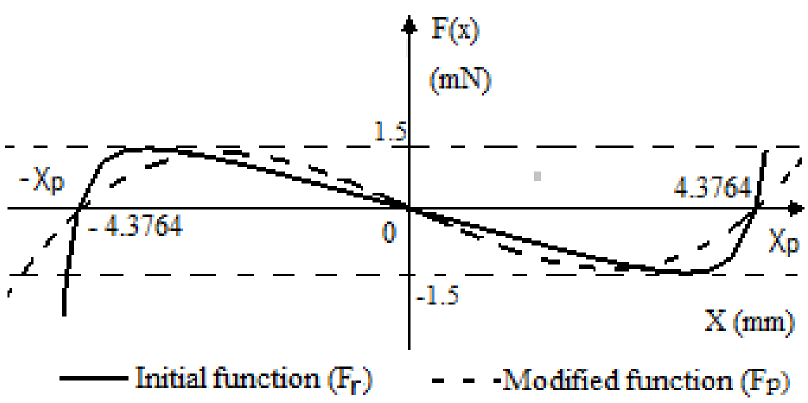

Fig. 2. The graph of (F_p, F_r) functions, with the same roots and the same intergraphic area

The time argument will become:

$$\alpha = \frac{0.93432}{\sqrt{x_0}} \tag{73}$$

Finally we obtain the function:

$$G_n(X_1) = x_{01}\ sn\left[\frac{0.93432}{\sqrt{x_0}}\ t,\ \frac{6.107}{\sqrt{2x_0^2 - 1}}\right] \tag{74}$$

Which is accompanied by the multi- graph as a function of t and initial position:

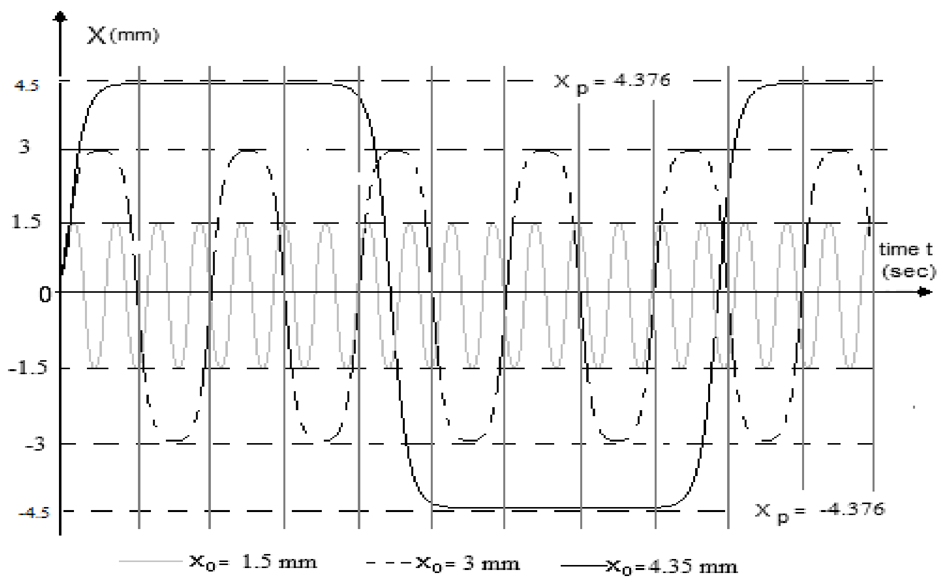

Fig. 3. Graph of G_n function, deppending on starting possition

c) The third 2nd order linear differential equation, in this form:

$$11.81\frac{d^2X_2(t)}{dt^2} + 2.334\quad X_2(t) = 0 \tag{75}$$

With proper pulsation:

$$\omega_2^2 = \frac{2.334}{11.81} \Leftrightarrow \omega_2 = \sqrt{\frac{2.334}{11.81}} = 0.444\ \text{rad/s} \tag{76}$$

And we get a trigonometric form of solutions:

$$X_2(t) = x_{02}\sin(0.444\quad t) \tag{77}$$

4 Discussion

In order to produce mechanical oscillations with "controlled halts", as their existence has been experimentally demonstrated, for the first time, by the authors and presented in Fig. 9, we have chosed the physical constants of the system, so that those two components in trigonometric sine, must have the same frequency of oscillation.

The component that oscillates after elliptic sine law, will have a variable frequency, which depends on the strength of the eccentricity of the nonlinear component. It can be adjusted in X_{01}, initial start position, so there is an equal condition of oscillation frequencies. The system naturally tends to self-synchronize, becoming stronger or weaker in eccentricity to "hold" in frequency.

In this way, the system will always tend to self- regulate in a natural way, the main oscillator will periodically enter in standby (stops), synchronizing with the frequency of the second oscillator, which requires in fact the repetition frequency of this process.

In order to get controlled halts it is indicated to produce a difference between the individual oscillations frequencies, by choosing the physical constants (masses and elastic constant of the springs).

The phenomenon will be simulated for that value for: x_{01}, in order to achieve $\alpha = \omega_1^2 = \omega_2^2 = 0.44$

$$\alpha = \frac{0.93432}{\sqrt{x_0}} = 0.446 \Leftrightarrow x_{01} = 4.3734 \tag{78}$$

So the start position for the main oscillator will have the value above.

We found a system with three decoupled solutions, explained as it follows:

$$V : \begin{cases} X_1(t) = x_{01} \sin(0.444 \ \ t) \\ G_n(X(t)) = x_{01} \ \ sn \left[\dfrac{0.93432}{\sqrt{x_{01}}} \ \ t, \ \ \dfrac{6.107}{\sqrt{2x_{01}^2 - 1}} \right] \\ X_2(t) = x_{02} \sin(0.444 \ \ t) \end{cases} \tag{79}$$

To reach the basic vector space solution, we perform multiplications. We will introduce them in a linear combination, using inverse vector transformation as follows, in order to obtain the final solution, which we have searched.

We will have to find the final solutions, as shown below:

$$E : \begin{pmatrix} X_1(t) \\ G_n(X(t)) \\ X_1(t) \end{pmatrix} = \begin{pmatrix} -0.585 & 2 & 1.334 \\ 2.9 & 2.085 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_{01} \sin(0.44 \ \ t) \\ x_{01} \ \ sn \left[\dfrac{0.93432}{\sqrt{x_0}} \ \ t, \ \ \dfrac{6.107}{\sqrt{2x_0^2 - 1}} \right] \\ x_{02} \sin(0.44 \ \ t) \end{pmatrix} \tag{80}$$

- We will return to the basic vector space for finding the general solution of this problem in the initial conditions.
- Since the nonlinear function $G_n(X_1)$ was artificially introduced in the vector space V, it has no physical significance in base vector space E and its corresponding equation will not be written.

$$\begin{cases} X_1(t) = -0.585x_{01}\sin(0.444\ t) + 2x_{01}\ sn\left[\dfrac{0.93432}{\sqrt{x_{01}}}\ t,\ \dfrac{6.107}{\sqrt{2x_{01}^2-1}}\right] + 1.334x_{02}\sin(0.444\ t) \\ X_2(t) = x_{01}\ sn\left[\dfrac{0.93432}{\sqrt{x_{01}}}\ t,\ \dfrac{6.107}{\sqrt{2x_{01}^2-1}}\right] - x_{02}\sin(0.444\ t) \end{cases} \quad (81)$$

4.1 We associate the most important, graphic representations, in the following:

The previous formula has the following graph, which includes the corresponding values of some initial perturbations applied to the two oscillators.

It was considered that these two oscillators were trained in motion at time t = 0, with zero initial speed.

a) For

$$\begin{cases} x_{01} = 4.3732\ mm \\ x_{02} = -1.0\ mm \end{cases} \quad (82)$$

We have the graph bellow:

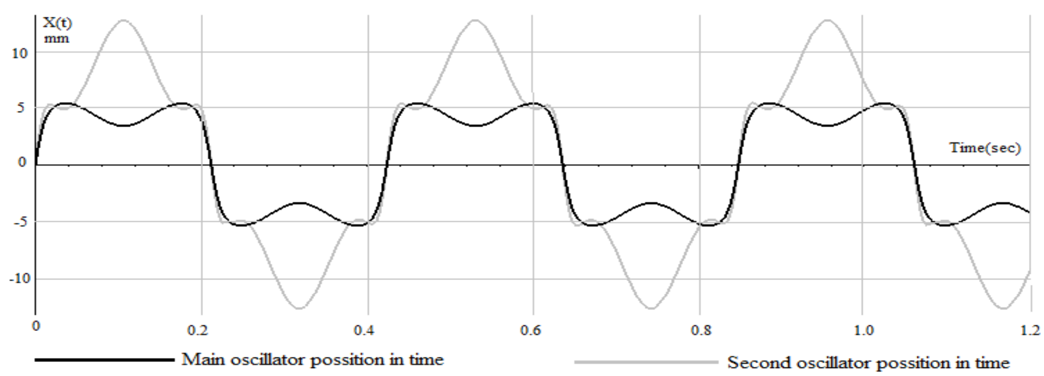

Fig. 4. Graph for the given physical system and starting conditions a

In the Fig. 4, are graphically represented the motion equations (81), as a function of time, for the initial conditions (82). The grey line represents the law of motion for the second oscillator and the black line represents the law of motion of the main oscillator.

b) For:

$$\begin{cases} x_{01}=4.3732\ mm \\ x_{02}=-0.1\ mm \end{cases} \quad (83)$$

We have the graph bellow:

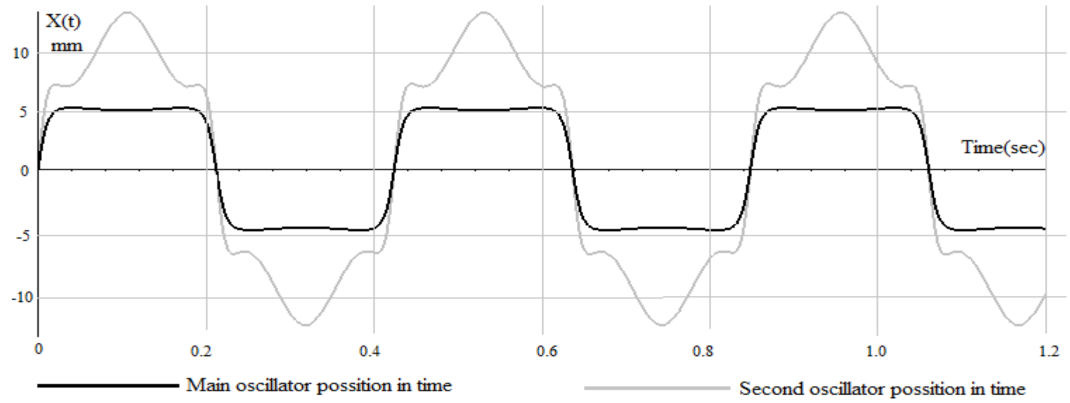

Fig. 5. Graph for given physical system and starting conditions b

In the upper graph, we see what is happen if we maintain the initial starting position of the main oscillator, and we modify only the starting positions of the secondary oscilator (10 times lower than the previous example). We obtain the changing of the graphic concavity of main oscilator, as can be seen.

c) For:

$$\begin{cases} x_{01} = 4.373 \ mm \\ x_{02} = 1.5 \ mm \end{cases} \quad (84)$$

If the starting position x_{02} is in the same direction with x_{01}, we obtain the next graph.

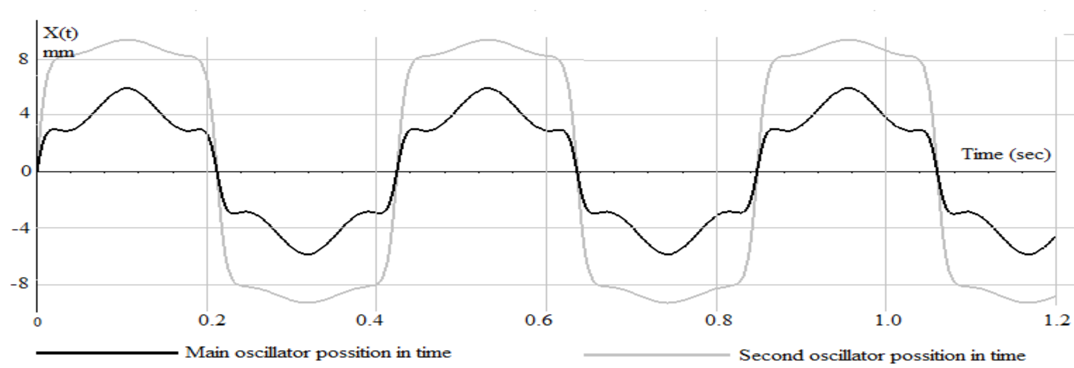

Fig. 6. Graph for given physical system and starting condition of item c

d) For:

$$\begin{cases} x_{01} = 4.373 \ mm \\ x_{02} = 1.95 \ mm \end{cases} \quad (85)$$

If the same value of x_{02} is more increasing, as it overpasses a certain limit, then we can notice that the subsystems change their roles, as is shown in the next figure. The second oscillator is halt for a period of time, while the main oscillator, make speediest movements, as it can be seen in the following:

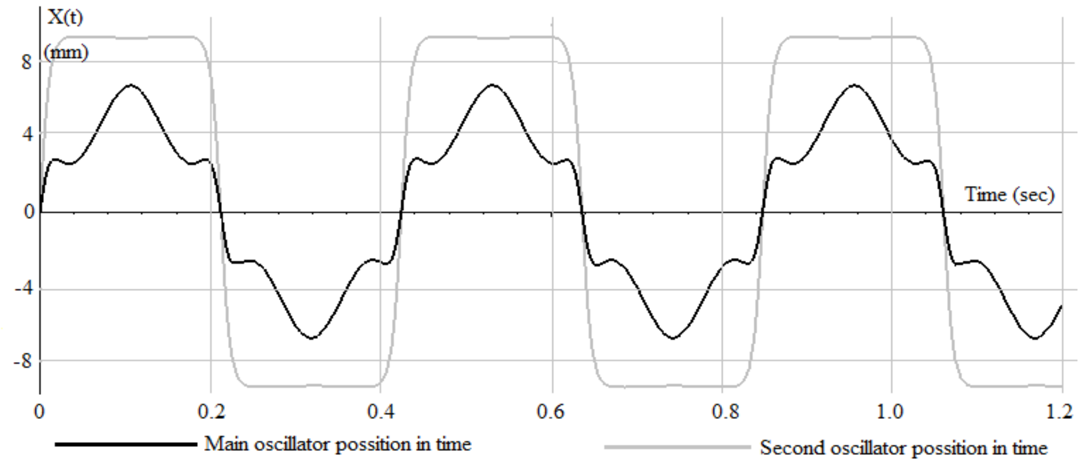

Fig. 7. Graph for given physical system and starting condition of item d

e) For:

$$\begin{cases} x_{01} = 4.373 \ mm \\ x_{02} = 2.3 \ mm \end{cases} \quad (86)$$

As we see, if the same value of x_{02} is more increasing, the graph concavity are now changing, like in the follow:

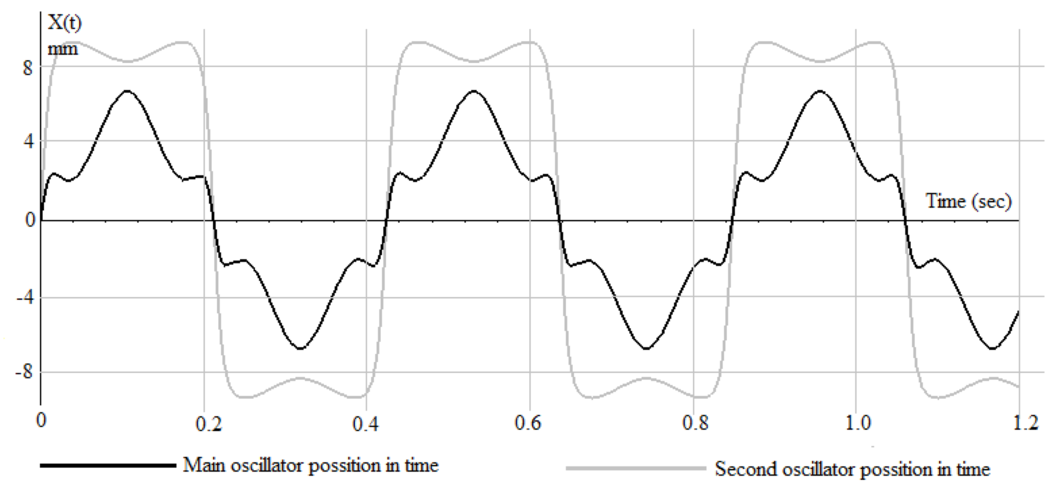

Fig. 8. Graph for given physical system and starting condition e

f) The theoretical study presented above is confirmed by next the experimental results.

Description of the experimental device, which also was conducted by the authors, but this is not the subject of this work, and will be treated in other scientific article, that will be a more complex model, treated for the experimental point of view, by considering of friction forces, disipative interactions between magnets, dipole - dipole interactions and excitation forces.

By reason, we present the graphic result of experimental test, ilustrating the mechanical system behavior, based on the numerical acquisition of data, and their further processing. The presented experimental situation is most similar with upper theoretical case a or b.

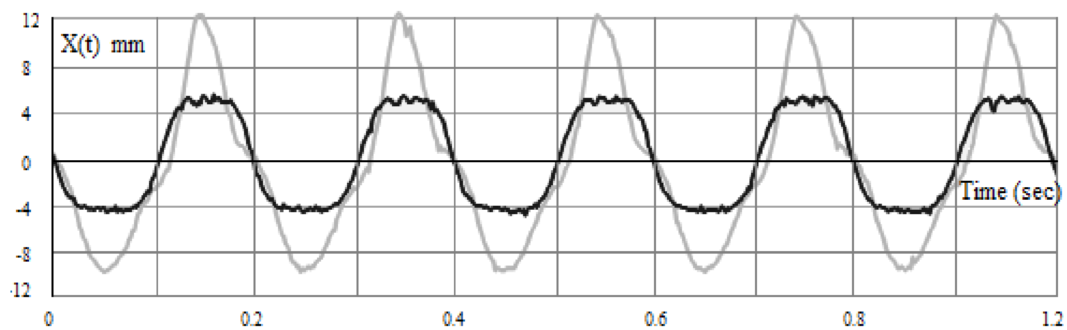

Fig. 9. Graph for experimental results

5 Conclusions

We have come to the following main conclusions:

- In the mechanical system presented in this research paper, the two oscillating subsystems do not produce beats phenomenon during their oscillations. Surprisingly, although both subsystems do not have the same individual frequency of oscillation, they synchronize and oscillate in phase, taking turns. In the end, a composition of the oscillations is to be found, in subtractive and additive ways.
- In terms of application, during the motion laws simulation, we have noticed that this system has an optimum performance, in the circumstances in which m_2 is at least the double of m_1.
- The oscillations which are presented in this paper, may also be obtained, in an artificial way, in mathematical terms, by composing the elliptic sine and trigonometric sine functions, under certain circumstances
- The halt time of main oscillator depends on the physical constants of the system, specifically on the mass difference between the first and the second subsystem.
- Those two subsystems could have different oscillatory frequencies and in this case the beats of oscillations can occur, while the component which oscillates after an elliptic sine law, will have a variable frequency, depending on the strength of eccentricity in the nonlinear field.
- Both subsystems naturally tend to self- synchronize, the nonlinear field of the main oscillator becoming stronger or weaker in elliptic eccentricity, for adjusting its frequency with the second.
- In fact, the mass difference between these two-coupled oscillating subsystems is offset by a more profound or superficial presence in a nonlinear field, because the subsystems 1 and 2 naturally tend to come into play and give each other time. “Controlled halts” are produced without mechanical collision, from a distance, by transferring mechanical momentum, through the common spring. We regularly transfer momentum, through spring connection, because during shutdown/startup, the secondary oscillator is accelerated, respectively decelerated.

- The motion equations include three components, two trigonometric equations and one elliptical equation. It may be said that these allow for the description of several other types of movements, not yet shown from an experimental point of view, but they were highlighted by the initial conditions (start) of the mechanical subsystems, and they were shown in the graphics.
- We specify that this phenomenon, which is discovered and presented in this study, has been used in the construction of a new type of mechanical engine, which is based on controlled halts of movement.
- By using special nonlinear transformations, we apply a method of algebraic vector transformation, applicable to linear systems in a nonlinear case, the stated analytical results being in accordance with experimental results.
- As this is a theoretical article, the physical experiment and the experimental data have not been described in detail, as exceeding the purposes of this paper.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Richard H. Enns, George C. McGuire. Nonlinear physics with Mathematica for scientist and engineers.
ISBN 0-8176-4223-4, 518-518.

[2] Nicuşor Nistor, Constantin Gheorghieş, Nelu Cazacu. The motion equation of a spring-magnet-mass system placed in nonlinear magnetic field. An analytical solution of elliptic sine form functions. British Journal of Mathematics & Computer Science.
DOI: 10.9734/BJMCS/2016/20083, SCIENCEDOMAIN International

[3] Todor Todorov, Rumen Nikolov. Dynamical modeling of mechanical oscillations in magnetic field Recent Journal. 2008;9(1):22.
ISSN: 1582-0246.

[4] Arnold VI. Problem in mathematical methods of classical mechanics. New York: Springer-Verlag; 1989.

[5] Temple H. Fay, Sarah Duncan Graham. Coupled spring equations. University of Southern Mississippi, Int. J. Math. Educ. Sci. Technol. 2003;34(1):65–79.

[6] Nicuşor Nistor, Constantin Gheorghieş, Nelu Cazacu. Algorithm to find the general laws of motion of the double oscillator, by using advance transformations on 1-parametric, vector space R^2. Annals of “Dunărea de Jos” University of Galaţi, Mathematics, Physics, Theoretical mechanics, Fascicle ii, year vii (xxxviii) 2015, special issue.

[7] Bandyrskii B, Lazurchak I, Makarov V, Sapagovas M. Eigenvalue problem for the second order differential equation with nonlocal conditions. Nonlinear Analysis: Modeling and Control. 2006;11(1):13–32.

[8] Jacobi CGJ. Fundamenta nova theoriae functionum ellipticarum (in Latin). Königsberg. Reprinted by Cambridge University Press. 1829; 2012.
ISBN: 978-1-108-05200-9.

[9] Abramowitz, Milton, Stegun, Irene A, eds. Handbook of mathematical functions with formulas, graphs and mathematical tables. New York: Dover. 1965; Chapter 16:569.
ISBN: 978-0486612720.

[10] Belendez C. Pascual, Mendez DI, Belendez T, Neipp C. Exact solution for the nonlinear pendulum. Revista Brasileira de Ensino de Fisica. 2007;29(4):645-648.

[11] Alvaro H. Salas, Jairo E. Castillo H. Exact solution to duffin equation and the pendulum equation. Applied Mathematical Sciences. 2014;8(176):8781–8789.

Peer-review history:
The peer review history for this paper can be accessed here:
http://sciencedomain.org/review-history/14572