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Research Article

Algebro-Geometric Solutions of a $(2+1)$-Dimensional Integrable Equation Associated with the Ablowitz-Kaup-Newell-Segur Soliton Hierarchy

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The $(2+1)$-dimensional Lax integrable equation is decomposed into solvable ordinary differential equations with the help of known $(1+1)$-dimensional soliton equations associated with the Ablowitz-Kaup-Newell-Segur soliton hierarchy. Then, based on the finite-order expansion of the Lax matrix, a hyperelliptic Riemann surface and Abel-Jacobi coordinates are introduced to straighten out the associated flows, from which the algebro-geometric solutions of the $(2+1)$-dimensional integrable equation are proposed by means of the Riemann θ functions.

1. Introduction

Algebro-geometric solutions are an important class among exact solutions of soliton equations, which can be regarded as explicit solutions of the nonlinear integrable evolution equation and used to approximate more general solutions. Based on the nonlinearization technique of Lax pairs and direct method, many of algebro-geometric solutions of $(1+1)$-dimensional [1–3], $(2+1)$-dimensional [4, 5], and differential-difference [5, 6] soliton equations have been obtained, such as the Gerdjikov-Ivanov, modified Kadomtsev-Petviashvili, and Toda lattice equations [7–9]. The existence of infinitely many exact solutions is a reflection of this complete integrability.

Many other techniques for finding exact solutions have been also discovered: inverse scattering theory, Darboux transformation, Riemann-Hilbert method, etc. Recently, more exact solutions of soliton equations are found [10–13], and more dynamic behaviors are studied [14–16].

Ablowitz-Kaup-Newell-Segur (AKNS) soliton hierarchy is an important class of integrable equations, which can be reduced to Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV), sine-Gordon equation hierarchies, etc. The purpose of the paper is to further develop the direct method for constructing algebro-geometric solution of the following $(2+1)$-dimensional integrable equation [15] which concerns with the AKNS soliton hierarchy [17].

$$\begin{cases} u_t = -\frac{1}{2} u_{xy} + u \partial_x^{-1} (uv)_y, \\ v_t = \frac{1}{2} v_{xy} - v \partial_x^{-1} (uv)_y. \end{cases} \tag{1}$$

In fact, system (1) is the Lax integrable equations from the AKNS soliton hierarchy, which has nonisospectral zero curvature representation. Bäcklund transformation for a splitting of $sl(2)$ and a soliton exact solution for it was obtained [18].

The whole paper is organized as follows: in Section 2, we use Lenard operator pairs to briefly derive $(1+1)$-dimensional AKNS soliton hierarchy and give the $(2+1)$-dimensional integrable equation (1). Then, in Section 3, based on the solutions of the $(1+1)$-dimensional soliton equations and the elliptic coordinates, the solution of the $(2+1)$-dimensional integrable equation is reduced to solving ordinary differential equations. In Section 4, a hyperelliptic Riemann surface and Abel-Jacobi coordinates are introduced to straighten the associated flows. The Jacobi's inversion problem is discussed, from which the algebro-geometric solution of the $(2+1)$-dimensional integrable equation is obtained in terms of the Riemann theta functions. A short summary is in Section 5.

2. The $(2+1)$-Dimensional Soliton Equation

It is well known that the AKNS soliton hierarchy is isospectral evolution equation hierarchy associated with the spectral problem [17].

$$\begin{gathered}\psi_x = U\psi = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}\psi, \\ \lambda_t = 0, \\ \psi = (\psi_1, \psi_2)^T. \end{gathered} \tag{2}$$

Consider the Lenard gradient sequence $\{S_j\}_{j=0}^{\infty}$ by

$$\begin{gathered} KS_{j-1} = JS_j, \\ S_j\big|_{(q,r)=(0,0)} = 0, \\ S_0 = (0,0,1)^T, \end{gathered} \tag{3}$$

where $S_j = (S_j^{(1)}, S_j^{(2)}, S_j^{(3)})$ and

$$\begin{gathered} K = \begin{pmatrix} \frac{1}{2}\partial & 0 & -r \\ 0 & -\frac{1}{2}\partial & -q \\ -q & r & \partial \end{pmatrix}, \\ J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -q & r & \partial \end{pmatrix}. \end{gathered} \tag{4}$$

It is easy to see that S_j is uniquely determined by the recursion relation. A direct calculation gives that

$$S_1 = \begin{pmatrix} -r \\ -q \\ 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} -\frac{1}{2}r_x \\ -\frac{1}{2}q_x \\ -\frac{1}{2}qr \end{pmatrix},$$

$$S_3 = \begin{pmatrix} \left(-\frac{1}{4}r_{xx} + \frac{1}{2}qr^2\right) \\ \left(-\frac{1}{4}q_{xx} + \frac{1}{2}q^2 r\right) \\ \frac{1}{4}(rq_x - qr_x) \end{pmatrix}. \tag{5}$$

The auxiliary spectral of (2) is

$$\psi_{tn} = V\psi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\psi = \begin{pmatrix} \sum_{j=0}^{n} S_j^{(3)}\lambda^{n-j} & \sum_{j=0}^{n} S_j^{(2)}\lambda^{n-j} \\ \sum_{j=0}^{n} S_j^{(1)}\lambda^{n-j} & -\sum_{j=0}^{n} S_j^{(3)}\lambda^{n-j} \end{pmatrix}\psi. \tag{6}$$

The compatibility condition between (2) and (6) is the zero curvature equation:

$$U_{tn} - V_x^{(n)} + \left[U, V^{(n)}\right] = 0, \tag{7}$$

which is equivalent to the hierarchy of soliton equations

$$X_n = \begin{pmatrix} q_{tn} \\ r_{tn} \end{pmatrix} = \begin{pmatrix} -2S_{n+1}^{(2)} \\ 2S_{n+1}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} S_{n+1}^{(1)} \\ S_{n+1}^{(2)} \end{pmatrix}, n = 1, 2,. \cdots \tag{8}$$

The first two nontrivial members in the hierarchy are

$$\begin{cases} q_{t_2} = \frac{1}{2}\left(q_{xx} - rq^2\right), \\ r_{t_2} = \frac{1}{2}\left(-r_{xx} + qr^2\right), \end{cases} \tag{9}$$

$$\begin{cases} q_{t_3} = \frac{1}{4}\left(-q_{xxx} + 6qrq_x\right), \\ r_{t_3} = \frac{1}{4}\left(r_{xxx} + 6rqr_x\right). \end{cases} \tag{10}$$

Let $t_2 = y$, $t_3 = t$, $u(x,y,t) = q(x,y,t)$, and $v(x,y,t) = r(x,y,t)$ in (9) and (10); then, we can obtain the $(2+1)$-dimensional equation (1) by the use of the following equation:

$$(uv_x - u_x v)_x = -2(uv)_y. \tag{11}$$

Therefore, if q and r are the compatible solutions of (9) and (10), then we can get that $u = q$ and $v = r$ are also the solutions of the $(2+1)$-dimensional equation (1).

3. Variable Separation

In this section, we shall show how the $(1+1)$-dimensional (9) and (10) are reduced to solvable ordinary differential equations. Assume that (2) and (6) have two basic solutions $\psi=(\psi_1,\psi_2)^T$ and $\phi=(\phi_1,\phi_2)^T$. We define a matrix W of three functions f, g, h by

$$W=\frac{1}{2}\left(\phi\psi^T+\psi\phi^T\right)\sigma=\begin{pmatrix} f & g \\ h & -f\end{pmatrix}, \sigma=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}. \tag{12}$$

It is easy to verify by (2) and (6) that

$$\begin{aligned} W_x &= [U, W], \\ W_{t_n} &= \left[V^{(n)}, W\right], \end{aligned} \tag{13}$$

which imply that the functions detW is a constant independent of x and t_m. Equation (13) can be written as

$$\begin{aligned} g_x &= -2g\lambda-2qf, \\ h_x &= 2h\lambda+2rf, \end{aligned} \tag{14}$$

$$\begin{aligned} f_x = qh-rg, \ g_t &= -2gA-2fB, \\ h_t &= 2hA+2fC, \\ f_t &= hB-gC. \end{aligned} \tag{15}$$

Now, suppose that the functions f, g, and h are finite-order polynomials in λ:

$$\begin{aligned} f &= \sum_{j=0}^{N+1} f_j\lambda^{N+1-j}, \\ g &= \sum_{j=0}^{N+1} g_j\lambda^{N+1-j}, \\ h &= \sum_{j=0}^{N+1} h_j\lambda^{N+1-j}. \end{aligned} \tag{16}$$

Substituting (16) into (14) yields

$$\begin{aligned} KG_{j-1} &= JG_j, \\ JG_0 &= 0, \\ KG_N &= 0, \\ G_j &= \left(g_j, h_j, f_j\right)^T. \end{aligned} \tag{17}$$

It is easy to see that $JG_0=0$ has the general solution:

$$G_0=\alpha_0 S_0, \tag{18}$$

where α_0 is constant of integration. So, $KerJ=\{cS_0|\forall c\}$. Acting with the operator $(J^{-1}K)^{K+1}$ upon (18), we can obtain from (3) and (17) that

$$G_k=\sum_{j=0}^{k}\alpha_j S_{k-j}, k=0,1,\cdots, \tag{19}$$

where $\alpha_0, \ldots, \alpha_k$ are integral constants. Substituting (19) into (17) obtains the following stationary evolution equation:

$$\alpha_0 KS_N+\cdots+\alpha_N KS_0=0. \tag{20}$$

This means that expression (16) is existent.

In what follows, we decompose (9) and (10) into systems of integrable ordinary differential equations. Without loss of generality, let $\alpha_0=1$. From (3) and (19), we have

$$\begin{aligned} g_0 &= 0, \\ g_1 &= -q, \\ g_2 &= \frac{1}{2}q_x-\alpha_1 q, \\ g_3 &= -\frac{1}{4}q_{xx}+\frac{1}{2}q^2r+\frac{\alpha_1}{2}q_x-\alpha_2 q, \end{aligned} \tag{21}$$

$$\begin{aligned} h_0 &= 0, \\ h_1 &= -r, \\ h_2 &= -\frac{1}{2}r_x-\alpha_1 r, \\ h_3 &= -\frac{1}{4}r_{xx}+\frac{1}{2}qr^2-\frac{\alpha_1}{2}r_x-\alpha_2 r, \end{aligned} \tag{22}$$

$$\begin{aligned} f_0 &= 1, \\ f_1 &= \alpha_1, \\ f_2 &= -\frac{1}{2}qr+\alpha_2, \end{aligned} \tag{23}$$

$$\begin{aligned} f_3 &= \frac{1}{4}\left(rq_x-qr_x\right)-\frac{\alpha_1}{2}qr+\alpha_3, g_4 \\ &= \frac{1}{8}q_{xxx}-\frac{3}{4}qrq_x-\frac{\alpha_1}{2}\left(\frac{1}{2}q_{xx}-q^2r\right)+\frac{\alpha_2}{2}q_x-\alpha_3 q, \end{aligned} \tag{24}$$

$$h_4=-\frac{1}{8}r_{xxx}+\frac{3}{4}qrr_x+\frac{\alpha_1}{2}\left(\frac{-1}{2}r_{xx}+r^2q\right)-\frac{\alpha_2}{2}r_x-\alpha_3 r, \cdots. \tag{25}$$

We can write g and h as the following finite products:

$$\begin{aligned} g &= -q\prod_{i=1}^{N}\left(\lambda-\mu_i\right), \\ h &= -r\prod_{i=1}^{N}\left(\lambda-\nu_i\right). \end{aligned} \tag{26}$$

Comparing the coefficients of λ^{N-1}, λ^{N-2}, and λ^{N-3}, we get

$$
\begin{aligned}
g_2 &= q\sum_{i=1}^{N}\mu_i,\\
h_2 &= r\sum_{i=1}^{N}\nu_i,\\
g_3 &= -q\sum_{i<j}\mu_i\mu_j,\\
h_3 &= -r\sum_{i<j}\nu_i\nu_j,\\
g_4 &= q\sum_{i<j<k}\mu_i\mu_j\mu_k,\\
h_4 &= r\sum_{i<j<k}\nu_i\nu_j\nu_k,
\end{aligned}
\tag{27}
$$

which together with (23), we obtain

$$
\begin{aligned}
\frac{1}{2}\partial_x \ln q &= \sum_{i=1}^{N}\mu_i + \alpha_1,\\
\frac{1}{2}\partial_x \ln r &= -\sum_{i=1}^{N}\nu_i - \alpha_1,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\frac{1}{2}\partial_y \ln q &= \sum_{i<j}\mu_i\mu_j + \alpha_1\sum_{i=1}^{N}\mu_i + {\alpha_1}^2 - \alpha_2,\\
\frac{1}{2}\partial_y \ln r &= -\sum_{i<j}\nu_i\nu_j - \alpha_1\sum_{i=1}^{N}\nu_i - {\alpha_1}^2 + \alpha_2,
\end{aligned}
\tag{29}
$$

$$
\frac{1}{2}\partial_t \ln q = -\sum_{i<j<k}\mu_i\mu_j\mu_k - \alpha_1\sum_{i<j}\mu_i\mu_j - \left({\alpha_1}^2 - \alpha_2\right)\sum_{i=1}^{N}\mu_i - {\alpha_1}^3 + 2\alpha_1\alpha_2 - \alpha_3,
\tag{30}
$$

$$
\frac{1}{2}\partial_t \ln r = \sum_{i<j<k}\nu_i\nu_j\nu_k + \alpha_1\sum_{i<j}\nu_i\nu_j + \left({\alpha_1}^2 - \alpha_2\right)\sum_{i=1}^{N}\nu_i + {\alpha_1}^3 - 2\alpha_1\alpha_2 + \alpha_3.
\tag{31}
$$

Let us consider the function detW, which is a $(2N+2)$-order polynomial in λ with constant coefficients of the x flow and t_n flow:

$$
-\det W = f^2 + gh = \prod_{j=1}^{2N+2}\left(\lambda - \lambda_j\right) \equiv R(\lambda).
\tag{32}
$$

Substituting (16) into (32), comparing the coefficient of λ^{2N+1}, λ^{2N}, and λ^{2N-1}, and considering (23), we can obtain

$$
\begin{aligned}
\alpha_1 &= -\frac{1}{2}\sum_{j=1}^{2N+2}\lambda_j,\\
\alpha_2 &= \frac{1}{2}\sum_{i<j}\lambda_i\lambda_j - \frac{1}{8}\left(\sum_{j=1}^{2N+2}\lambda_j\right)^2,\\
\alpha_3 &= \frac{1}{2}\sum_{i<j<k}\lambda_i\lambda_j\lambda_k + \frac{1}{4}\sum_{j=1}^{2N+2}\lambda_j\sum_{i<j}\lambda_i\lambda_j - \frac{1}{16}\left(\sum_{j=1}^{2N+2}\lambda_j\right)^3.
\end{aligned}
\tag{33}
$$

From (32) we see that

$$
\begin{aligned}
f|_{\lambda=\mu_k} &= \sqrt{R(\mu_k)},\\
f|_{\lambda=\nu_k} &= \sqrt{R(\nu_k)}.
\end{aligned}
\tag{34}
$$

Using (14) and (26), we get

$$
\begin{aligned}
g_x\big|_{\lambda=\mu_k} &= -2qf\big|_{\lambda=\mu_k} = -q\mu_{kx}\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i),\\
h_x\big|_{\lambda=\nu_k} &= 2rf\big|_{\lambda=\nu_k} = -r\nu_{kx}\prod_{i=1,i\neq k}^{N}(\nu_k - \nu_i).
\end{aligned}
\tag{35}
$$

Together with (34), we get

$$
\begin{aligned}
\mu_{kx} &= \frac{2\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i)},\\
\nu_{kx} &= \frac{-2\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N}(\nu_k - \nu_i)}.
\end{aligned}
\tag{36}
$$

Similarly, using (6) ($n = 2$, $n = 3$), (16), (26), and (34), we get

$$
\begin{aligned}
\mu_{kt_2} = \frac{2\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i)}\Bigg(&-\mu_k^2 + \left(\sum_{i=1}^{N}\mu_i + \alpha_1\right)\mu_k - \sum_{i<j}\mu_i\mu_j\\
&- \alpha_1\sum_{i=1}^{N}\mu_i - \alpha_1^2 + \alpha_2\Bigg),
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\nu_{kt_2} = \frac{-2\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N}(\nu_k - \nu_i)}\Bigg(&-\nu_k^2 + \left(\sum_{i=1}^{N}\nu_i + \alpha_1\right)\nu_k - \sum_{i<j}\nu_i\nu_j\\
&- \alpha_1\sum_{i=1}^{N}\nu_i - \alpha_1^2 + \alpha_2\Bigg),
\end{aligned}
\tag{38}
$$

$$\mu_{kt_3} = \frac{2\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)}\left(-\mu_k^3+\left(\sum_{i=1}^{N}\mu_i+\alpha_1\right)\mu_k^2-\left(\sum_{i<j}\mu_i\mu_j+\alpha_1\sum_{i=1}^{N}\mu_i+\alpha_1^2-\alpha_2\right)\mu_k \right.$$
$$\left. +\sum_{i<j<k}\mu_i\mu_j\mu_k+\alpha_1\sum_{i<j}\mu_i\mu_j+\left(\alpha_1^2-\alpha_2\right)\sum_{i=1}^{N}\mu_i+\alpha_1^3-2\alpha_1\alpha_2+\alpha_3\right). \tag{39}$$

Therefore, if $\lambda_1,\cdots,\lambda_{2N+2}$ are $2N+2$ distinct parameters and $\mu_k,\nu_k(k=1,\cdots,N)$ are compatible solutions of differential equations (36), (37), and (39), then q and r determined by (28) are the compatible solution of (9) and (10), so we can get that u and v are also the solution of the $(2+1)$-dimensional equation (1).

4. Algebro-Geometric Solution

We first introduce the hyperelliptic Riemann surface

$$\begin{aligned}\Gamma &: \xi^2 = R(\lambda),\\ R(\lambda) &= \prod_{j=1}^{2N+2}\left(\lambda-\lambda_j\right),\end{aligned} \tag{40}$$

with genus $g=N$. On Γ, there are two infinite points ∞_1 and ∞_2, which are not branch points of Γ. Equip Γ with the canonical basis of cycles $a_1,\cdots,a_N,b_1,\cdots,b_N$, and the holomorphic differentials

$$\tilde{\omega}_l = \frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}},\ l=1,2,\cdots,N. \tag{41}$$

Then, the period matrices A and B are defined by

$$\begin{aligned}A_{ij} &= \int_{a_j}\tilde{\omega}_i,\\ B_{ij} &= \int_{b_j}\tilde{\omega}_i.\end{aligned} \tag{42}$$

Using A and B, we can define the matrices C and τ, where

$$\begin{aligned}C &= \left(C_{ij}\right) = A^{-1},\\ \tau &= \left(\tau_{ij}\right) = CB = A^{-1}B.\end{aligned} \tag{43}$$

Then, matrix τ can be shown to be symmetric, and it has positive define imaginary part. We normalize $\widetilde{\omega_j}$ into the new basis ω_j:

$$\omega_j = \sum_{l=1}^{N}C_{jl}\widetilde{\omega_l},\ l=1,2,\cdots,N. \tag{44}$$

which satisfy

$$\begin{aligned}\int_{a_k}\omega_i &= \sum_{l=1}^{N}C_{jl}\int_{a_k}\widetilde{\omega_l} = \sum_{l=1}^{N}C_{jl}A_{lk} = \delta_{jk},\\ \int_{b_k}\omega_i &= \sum_{l=1}^{N}C_{jl}\int_{b_k}\widetilde{\omega_l} = \sum_{l=1}^{N}C_{jl}B_{lk} = \tau_{jk}.\end{aligned} \tag{45}$$

For a fixed point p_0, then we introduce Abel-Jacobi coordinate as follows:

$$\rho_m = \left(\rho_m^{(1)},\rho_m^{(2)},\cdots,\rho_m^{(N)}\right)^T,\ m=1,2, \tag{46}$$

whose components are

$$\rho_1^{(j)}(x,y,t) = \sum_{k=1}^{N}\int_{p_0}^{\mu_k(x,y,t)}\omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N}\int_{p_0}^{\mu_k(x,y,t)}C_{jl}\frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}}, \tag{47}$$

$$\rho_2^{(j)}(x,y,t) = \sum_{k=1}^{N}\int_{p_0}^{\nu_k(x,y,t)}\omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N}\int_{p_0}^{\nu_k(x,y,t)}C_{jl}\frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}}. \tag{48}$$

From (47) and the fist expression of (36), we get

$$\begin{aligned}\partial_x\rho_1^{(j)} &= \sum_{k=1}^{N}\sum_{l=1}^{N}C_{jl}\frac{\mu_k^{l-1}\mu_{kx}}{\sqrt{R(\mu_k)}} = \sum_{k=1}^{N}\sum_{l=1}^{N}C_{jl}\frac{\mu_k^{l-1}}{\prod_{i\neq k}\left(\mu_k-\mu_j\right)}\\ &= 2C_{jN} = \Omega_0^{(j)},\ j=1,\cdots,N,\end{aligned} \tag{49}$$

with the help of the following equality

$$\sum_{k=1}^{N}\frac{\mu_k^{l-1}}{\prod_{i=i,i\neq k}^{N}(\mu_k-\mu_i)} = \delta_{lN},\ l=1,\cdots,N. \tag{50}$$

In a similar way, we obtain from (36)–(39), (47), and (48) that

$$\begin{aligned}\partial_y\rho_1^{(j)} &= 2\left(-C_{j,N-2}+\alpha_1C_{j,N-1}-\left(\alpha_1^2-\alpha_2\right)C_{j,N}\right) = \Omega_1^{(j)},\\ \partial_t\rho_1^{(j)} &= 2\left(-C_{j,N-3}+\alpha_1C_{j,N-2}-\left(\alpha_1^2-\alpha_2\right)C_{j,N-1}+\left(\alpha_1^3-2\alpha_1\alpha_2+\alpha_3\right)C_{j,N}\right) = \Omega_2^{(j)},\\ \partial_x\rho_2^{(j)} &= -\Omega_0^{(j)},\\ \partial_y\rho_2^{(j)} &= -\Omega_1^{(j)},\\ \partial_t\rho_2^{(j)} &= -\Omega_2^{(j)}.\end{aligned} \tag{51}$$

On the basis of these results, we get the following:

$$\begin{aligned}\rho_1^{(j)}(x,y,t) &= \Omega_0^{(j)}x+\Omega_1^{(j)}y+\Omega_2^{(j)}t+\gamma_0^{(j)},\\ \rho_2^{(j)}(x,y,t) &= -\Omega_0^{(j)}x-\Omega_1^{(j)}y-\Omega_2^{(j)}t+\gamma_1^{(j)},\end{aligned} \tag{52}$$

where

$$\gamma_0^{(j)} = \sum_{k=1}^{N} \int_{p_0}^{\mu_k(0,0,0)} \omega_j,$$
$$\gamma_1^{(j)} = \sum_{k=1}^{N} \int_{p_0}^{\nu_k(0,0,0)} \omega_j. \tag{53}$$

An Abel map on Γ is defined as

$$A(p) = \int_{p_0}^{p} \omega, \omega = (\omega_1,\cdots,\omega_N)^T,$$
$$A\left(\sum n_k p_k\right) = \sum n_k A(p_k). \tag{54}$$

Consider two special divisors $\sum_{k=1}^{N} p_m^{(k)} (m = 1, 2)$, and we have

$$A\left(\sum_{k=1}^{N} p_1^{(k)}\right) = \sum_{k=1}^{N} A\left(p_1^{(k)}\right) = \sum_{k=1}^{N} \int_{p_0}^{\widetilde{\mu_k}} \omega = \rho_1,$$
$$A\left(\sum_{k=1}^{N} p_2^{(k)}\right) = \sum_{k=1}^{N} A\left(p_2^{(k)}\right) = \sum_{k=1}^{N} \int_{p_0}^{\widetilde{\nu_k}} \omega = \rho_2, \tag{55}$$

where $p_1^{(k)} = (\widetilde{\mu_k}, \xi(\widetilde{\mu_k})), p_2^{(k)} = (\widetilde{\mu_k}, \xi(\widetilde{\mu_k}))$. The Riemann theta function of Γ is defined as

$$\theta(\zeta) = \sum_{z \in Z^N} \exp\left(\pi i \langle \tau z, z\rangle + 2\pi i \langle \zeta, z\rangle\right), \zeta \in C^N, \tag{56}$$

where $\zeta = (\zeta_1,\cdots,\zeta_N)^T, \langle \zeta, z\rangle = \sum_{j=1}^{N} \zeta_j z_j$. According to the Riemann theorem, there exist two constant vector $M_1, M_2 \in C^N$ such that

$$F_m = \theta(A(p) - \rho_m - M_m), m = 1, 2, \tag{57}$$

has exactly zeros at $\mu_1, \cdots, \mu_N$ for $m = 1$ or $\nu_1, \cdots, \nu_N$ for $m = 2$ and $m = 3$. To make the function single valued, the surface Γ is cut along all a_k, b_k to form a simple connected region, whose boundary is denoted by γ. Notice the fact that the integrals

$$\frac{1}{2\pi i} \int_{\gamma} \lambda^k d\ln F_m(\lambda) = I_k(\Gamma), k \geq 1, \tag{58}$$

are constants independent of ρ_1, ρ_2 with $I = I(\Gamma) = \sum_{j=1}^{N} \int_{a_j} \lambda^k \omega_j$. By the residue theorem, we have

$$I_k(\Gamma) = \sum_{l=1}^{N} \mu_l^k + \sum_{s=1}^{2} \operatorname{Re} s_{\lambda=\infty_s} \lambda^k d\ln F_1(\lambda), \tag{59}$$

$$I_k(\Gamma) = \sum_{l=1}^{N} \nu_l^k + \sum_{s=1}^{2} \operatorname{Re} s_{\lambda=\infty_s} \lambda^k d\ln F_2(\lambda). \tag{60}$$

Here, we only need to compute the residues in (59) for $k = 1, 2, 3$. In the way similar to calculations in [1, 2, 4], we obtain

$$\operatorname{Re} s_{\lambda=\infty_s} \lambda d\ln F_m(\lambda) = \operatorname{Re} s_{z=0} z^{-1} d\ln F_m\left(z^{-1}\right) = (-1)^s \partial_x \ln \theta_s^{(m)}, s = 1, 2\,; m = 1, 2. \tag{61}$$

where $\theta_s^{(1)} = \theta(\Omega_0 x + \Omega_1 y + \Omega_2 t + \pi_s), \theta_s^{(2)} = \theta(-\Omega_0 x - \Omega_1 y - \Omega_2 t + \eta_s)$, and π_s, η_s are constants. Thus from, we arrive at

$$\sum_{j=1}^{N} \mu_j = I_1 - \partial_x \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}},$$
$$\sum_{j=1}^{N} \nu_j = I_1 - \partial_x \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}. \tag{62}$$

Similarly, we obtain

$$\sum_{j=1}^{N} \mu_j^2 = I_2 + \frac{1}{2} \partial_y \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{2} \partial_x^2 \ln \theta_1^{(1)} \theta_2^{(1)},$$
$$\sum_{j=1}^{N} \nu_j^2 = I_2 + \frac{1}{2} \partial_y \ln \frac{\theta_2^{(2)}}{\theta_1^{(2)}} - \frac{1}{2} \partial_x^2 \ln \theta_1^{(1)} \theta_2^{(1)}, \tag{63}$$

Then, we can get

$$\partial_x \ln q = 2\left(I_1 - \partial_x \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right) + 2\alpha_1 = \Theta_1,$$

$$\partial_x \ln r = -2\left(I_1 - \partial_x \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}\right) - 2\alpha_1 = \Lambda_1,$$

$$\partial_y \ln q = \left(I_1 - \partial_x \ln \frac{\theta_1^{(1)}}{\theta_2^{(1)}}\right)^2 - \left(I_2 + \frac{1}{2} \partial_y \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{2} \partial_x^2 \ln\ln \theta_1^{(1)} \theta_2^{(1)}\right) + 2\alpha_1 \left(I_1 - \partial_x \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right) + 2\left(\alpha_1^2 + \alpha_2\right) = \Theta_2,$$

$$\partial_y \ln r = -\left(I_1 - \partial_x \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}\right)^2 + \left(I_2 + \frac{1}{2} \partial_y \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{2} \partial_x^2 \ln \theta_1^{(1)} \theta_2^{(1)}\right) - 2\alpha_1 \left(\left(I_1 - \partial_x \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}\right) - 2\left(\alpha_1^2 + \alpha_2\right) = \Lambda_2,$$

$$
\begin{aligned}
\partial_t \ln q = & -\Bigg(\frac{1}{3}\left(I_1 - \partial_x \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right)^3 - \left(I_2 + \frac{1}{2}\partial_y \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{2}\partial_x^2 \ln \theta_1^{(1)}\theta_2^{(1)}\right) \\
& \cdot \left(I_1 - \partial_x \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right) + \frac{2}{3}\Bigg(I_3 - \frac{1}{4}\partial_t \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{4}\partial_y^2 \ln \theta_1^{(1)}\theta_2^{(1)} \\
& + \frac{1}{4}\left(\partial_x \ln \theta_1^{(1)}\right)^3 + \frac{1}{4}\left(\partial_x \ln \theta_2^{(1)}\right)^3\Bigg) + \alpha_1 \Bigg(\left(I_1 - \partial_x \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right)^2 \\
& - \left(I_2 + \frac{1}{2}\partial_y \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{2}\partial_x^2 \ln \theta_1^{(1)}\theta_2^{(1)}\right)\Bigg) + 2\left(\alpha_1^2 - \alpha_2\right) \\
& \cdot \left(I_1 - \partial_x \ln \frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right)^3 + 2\left(\alpha_1^3 - 2\alpha_1\alpha_2 + \alpha_3\right)\Bigg) = \Theta_3,
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \ln r = & \frac{1}{3}\left(I_1 - \partial_x \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}\right)^3\Bigg) - \left(I_2 + \frac{1}{2}\partial_y \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}} - \frac{1}{2}\partial_x^2 \ln \theta_1^{(1)}\theta_2^{(1)}\right) \\
& \cdot \left(I_1 - \partial_x \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}\right) + \frac{2}{3}\Bigg(I_3 - \frac{1}{4}\partial_t \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}} + \frac{1}{4}\partial_y^2 \ln \theta_1^{(2)}\theta_2^{(2)} \\
& + \frac{1}{4}\left(\partial_x \ln \theta_1^{(2)}\right)^3 + \frac{1}{4}\left(\partial_x \ln \theta_2^{(2)}\right)^3\Bigg) - \alpha_1 \Bigg(\left(I_1 - \partial_x \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}\right)^2 \\
& - \left(I_2 + \frac{1}{2}\partial_y \ln \frac{\theta_2^{(2)}}{\theta_1^{(2)}} - \frac{1}{2}\partial_x^2 \ln \theta_1^{(1)}\theta_2^{(1)}\right) \\
& - 2\left(\alpha_1^2 - \alpha_2\right)\left(I_1 - \partial_x \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}\right)^3 - 2\left(\alpha_1^3 - 2\alpha_1\alpha_2 + \alpha_3\right)\Bigg) = \Lambda_3.
\end{aligned}
\tag{64}
$$

With the help of the above equations, we arrive at the algebro-geometric solution of the $(2+1)$-dimensional equation (1):

$$
\begin{aligned}
u = q = \exp\left(\int_{(0,0,0)}^{(x,y,t)} \Theta_1 dx + \Theta_2 dy + \Theta_3 dt + c_1\right), \\
v = r = \exp\left(\int_{(0,0,0)}^{(x,y,t)} \Lambda_1 dx + \Lambda_2 dy + \Lambda_3 dt + c_2\right),
\end{aligned}
\tag{65}
$$

where c_1 and c_2 are constants.

5. Summary

The nonisospectral $(2+1)$-dimensional breaking soliton system is given by the Lenard gradient sequence for a classical $(1+1)$-dimensional AKNS spectral problem. Then, the $(2+1)$-dimensional Lax integrable equation associated with the AKNS soliton hierarchy (1) is decomposed into solvable ordinary differential equations with the help of known $(1+1)$-dimensional soliton equations. With introducing the hyperelliptic Riemann surface and the Abel-Jacobi coordinates, the flow can be straighten out, and the algebro-geometric solutions of the $(2+1)$-dimensional soliton system (1) are presented by means of the Riemann θ functions.

Data Availability

All data and models generated or used during this study appear in the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References

[1] H. H. Dai and E. G. Fan, “Variable separation and algebro-geometric solutions of the Gerdjikov-Ivanov equation,” *Chaos, Solitons and Fractals*, vol. 22, no. 1, pp. 93–101, 2004.

[2] Y. J. Sun, Z. Wang, and H. Q. Zhang, “Algebro-geometric solutions of the Fokas-Lenells equation,” *Chinese Annals of Mathematics*, vol. 33, pp. 135–148, 2012.

[3] Y. J. Sun, Q. Ding, and H. Q. Zhang, “Algebro-geometric solutions of the D-ANKS equation,” *Acta Mathematica Scientia*, vol. 33, pp. 276–284, 2013.

[4] X. G. Geng and H. H. Dai, “Algebro-geometric solutions of (2+1)-dimensional coupled modified Kadomtsev-Petviashvili equations,” *Journal of Mathematical Physics*, vol. 41, no. 1, pp. 337–348, 2000.

[5] J. Wang, “Algebro-geometric solutions for some (2+1)-dimensional discrete systems,” *Nonlinear Analysis: Real World Applications*, vol. 9, no. 5, pp. 1837–1850, 2008.

[6] Y. F. Zhang and X. Z. Zhang, “Two kinds of discrete integrable hierarchies of evolution equations and some algebraic-geometric solutions,” *Advances in Difference Equations*, vol. 72, 18 pages, 2017.

[7] C. W. Cao, Y. T. Wu, and X. G. Geng, “Relation between the Kadometsev-Petviashvili equation and the confocal involutive system,” *Journal of Mathematical Physics*, vol. 40, no. 8, pp. 3948–3970, 1999.

[8] H. Y. Wei and G. M. Pi, “The Hamiltonian structures and algebro-geometric solution of the generalized Kaup-Newell soliton equations,” *Chinese Quarterly Journal of Mathematics*, vol. 34, pp. 209–220, 2019.

[9] C. Yue and T. C. Xia, “Algebro-geometric solutions of the coupled Chaffee-Infante reaction diffusion hierarchy,” *Advances in Mathematical Physics*, vol. 2021, Article ID 6618932, 21 pages, 2021.

[10] S. Kumar and S. K. Dhiman, “Lie symmetry analysis, optimal system, exact solutions and dynamics of solitons of a ($$3+1$$)-dimensional generalised BKP–Boussinesq equation,” *Pramana-Journal of Physics*, vol. 96, no. 1, pp. 1–20, 2022.

[11] B. B. Hu, J. Lin, and L. Zhang, “On the Riemann-Hilbert problem for the integrable three-coupled Hirota system with a 4×4 matrix Lax pair,” *Applied Mathematics and Computation*, vol. 428, article 127202, 2022.

[12] S. Kumar, A. Kumar, and H. Kharbanda, “Lie symmetry analysis and generalized invariant solutions of (2+1)-dimensional

dispersive long wave (DLW) equations," *Physica Scripta*, vol. 95, no. 6, article 065207, 2020.

[13] J. W. Wu, J. T. He, and J. Lin, "Nonlocal symmetries and new interaction waves of the variable-coefficient modified Korteweg–de Vries equation in fluid-filled elastic tubes," *The European Physical Journal Plus*, vol. 137, no. 7, pp. 1–7, 2022.

[14] B. B. Hu, J. Lin, and L. Zhang, "Dynamic behaviors of soliton solutions for a three-coupled Lakshmanan-Porsezian-Daniel model," *Nonlinear Dynamics*, vol. 107, pp. 2773–2785, 2022.

[15] S. Kumar and A. Kumar, "Dynamical behaviors and abundant optical soliton solutions of the cold bosonic atoms in a zig-Zag optical lattice model using two integral schemes," *Mathematics and Computers in Simulation*, vol. 201, pp. 254–274, 2022.

[16] A. A. Hendi, L. Ouahid, S. Kumar, S. Owyed, and M. A. Abdou, "Dynamical behaviors of various optical soliton solutions for the Fokas–Lenells equation," *Modern Physics Letters B*, vol. 35, no. 34, p. 2150529, 2021.

[17] G. Z. Tu, ""the trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems, "*journal of mathematical*," *Physics*, vol. 30, pp. 330–339, 1989.

[18] N. Lv, *Symmetry Reductions and Constructive Methods of Nonlinear Evolution Equations*, Dalian University of Technology, Dalian, 2012.