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Research Article

Modified Homotopy Perturbation Method and Approximate Solutions to a Class of Local Fractional Integrodifferential Equations

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In this paper, the local fractional version of homotopy perturbation method (HPM) is established for a new class of local fractional integral-differential equation (IDE). With the embedded homotopy parameter monotonously changing from 0 to 1, the special easy-to-solve fractional problem continuously deforms to the class of local fractional IDE. As a concrete example, an explicit and exact Mittag–Leffler function solution of one special case of the local fractional IDE is obtained. In the process of solving, two initial solutions are selected for the iterative operation of local fractional HPM. One of the initial solutions has a critical condition of convergence and divergence related to the fractional order, and the other converges directly to the real solution. This paper reveals that whether the sequence of approximate solutions generated by the iteration of local fractional HPM can approach the real solution depends on the selection of the initial approximate solutions and sometimes also depends on the fractional order of the selected initial approximate solutions or the considered equations.

1. Introduction

Fractals, solitons, and chaos together constitute three important branches of nonlinear sciences. In fractal space, there exist some magical functions which are continuous everywhere but nondifferentiable everywhere. The local fractional calculus [1] developed in recent years provides a powerful mathematical tool to handling with such type of nondifferentiable functions. Fractional calculus, which is widely believed to have originated more than 300 years ago, has attracted much attention [2–17]. It is of theoretical and practical value to solve fractional differential equations (DEs) directly connecting with fractional dynamical processes in a great many fields. For this reason, people often construct exact solutions of fractional DEs to obtain useful clues in these fractional dynamical processes for specific applications.

With the development of fractional calculus, many numerical and analytical methods for fractional DEs have been developed, such as integral transform method [1], series expansion method [3], Adomian decomposition method [4], Fan subequation method [5], variational iteration method (VIM) [6], variable separation method [7], finite difference method [8], homotopy perturbation method (HPM) [9], combined the HPM with Laplace transform [10], exp-function method [11], and Hirota bilinear method [12]. The HPM proposed by He [18] couples the homotopy method and the perturbation technique, which needs no the small parameters embedded in differential equations. More importantly, it is indicated in [19] that the HPM can truly eliminate the limitations existing in traditional perturbation methods.

One of the advantages of local fractional derivative is that it has been successfully used to describe some nondifferential problems appearing in science and engineering [1]. The concept of local fractional derivative, which is based on Riemann-Liouville fractional derivative, can be retrospected to Kolwankar and Gangal's pioneering work [13]. More specifically, if the following limit exists and is finite for a given continuous function $u(x)$: $[0,1] \longrightarrow R$,

$$D^{\alpha}u(x_0) = \lim_{x\longrightarrow x_0} \frac{d^q(u(x)-u(x_0))}{d(x-x_0)^q}, (0<q<1), \quad (1)$$

then $D^{\alpha}u(x_0)$ is called q-order fractional derivative of $u(x)$ at the point $x = x_0$. Later, inspired by the relation $d^{\alpha}u(x) = \Gamma(1+\alpha)du(x)$ of Jumarie [20], the local fractional derivative of a local fractional continuous but nondifferentiable function $u(x)$ is defined as another form ([4], see Definition 1). Recently, the theory of local fractional calculus has been significantly developed. Yang and his collaborators [1, 3, 17] have made a series of achievements in the development of local fractional calculus. Benefiting from the graceful properties of local fractional calculus, some existing methods like those [5, 11, 12], originally proposed for DEs with integer orders, have successfully been extended to fractional DEs and many methods are meeting more and more challenges for solving fractional DEs.

The paper is aimed at establishing the local fractional HPM for a new class of local fractional IDEs:

$$\frac{d^{\alpha}u(x)}{dx^{\alpha}} - f(x) - \frac{1}{\Gamma(1+\alpha)}\int_0^1 g(x,t)u(t)(dt)^{\alpha} = 0, \quad (2)$$

where $0<\alpha\leq 1$, $u(x)$, $f(x)$, and $g(x,t)$ are the local fractional continuous but nondifferentiable functions, $d^{\alpha}u(x)/dx^{\alpha}$ and $(1/\Gamma(1+\alpha))\int_0^1 g(x,t)u(t)(dt)^{\alpha}$ represent the local fractional derivative and integral [1], respectively, and $\Gamma(1+\alpha)$ is the well-known Euler's Gamma function:

$$\Gamma(1+\alpha) = \int_0^{\infty} t^{\alpha}e^{-t}dt. \quad (3)$$

Considering a concrete application of the established local fractional HPM, we would like to solve a special case of Equation (2):

$$\frac{d^{\alpha}u(x)}{dx^{\alpha}} - 3E_{\alpha}(3x^{\alpha}) + \left(\frac{E_{\alpha}(3)}{\Gamma(1+\alpha)} - \frac{E_{\alpha}(3)-1}{3}\right)x^{\alpha} - \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{3x^{\alpha}t^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha} = 0, \quad (4)$$

where the Mittag–Leffler function

$$E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, \quad (5)$$

which is defined on a fractal set [1].

The organization of the rest of this paper is as follows. Section 2 recalls the local fractional derivative and integral and some basic properties. Section 3 establishes the local fractional HPM for the class of local fractional IDEs (2). Section 4 takes the local fractional IDE (4), a special case of Equation (2), to test the established local fractional HPM and discuss the influence of not only the initial approximate solutions but also the fractional order on whether the sequence of approximate solutions can approach the real solution. Section 5 employs He-Laplace method coupling the HPM with Laplace transform to solve the local fractional IDE (4) and compares the obtained results. Section 6 summarizes the whole paper.

2. Local Fractional Derivative and Integral and Some Basic Properties

In this section, we recall the local fractional derivative and integral and some basic properties.

Definition 1 (see [1]). Let $u(x)$ be a local fractional continuous but nondifferentiable function; then, α-order local fractional derivative of $u(x)$ at the point $x = x_0$ reads

$$D_x^{\alpha}u(x_0) = \left.\frac{d^{\alpha}u(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x\longrightarrow x_0}\frac{\Delta^{\alpha}(u(x)-u(x_0))}{(x-x_0)^{\alpha}}, (0<\alpha\leq 1), \quad (6)$$

where $\Delta^{\alpha}(u(x)-u(x_0)) \cong \Gamma(1+\alpha)(u(x)-u(x_0))$.

The local fractional derivative has some basic properties [1]:

$$\begin{aligned}
&D_x^{\alpha}(\lambda u(x)+\mu v(x)) = \lambda D_x^{\alpha}u(x) + \mu D_x^{\alpha}v(x),\\
&D_x^{\alpha}(u(x)v(x)) = (D_x^{\alpha}u(x))v(x) + u(x)(D_x^{\alpha}v(x)),\\
&D_x^{\alpha}\frac{u(x)}{v(x)} = \frac{D_x^{\alpha}u(x)}{v(x)} - \frac{u(x)(D_x^{\alpha}v(x))}{v^2(x)},\\
&D_x^{\alpha}(C) = 0,\\
&D_x^{\alpha}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)} = \frac{x^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)},\\
&D_x^{\alpha}E_{\alpha}(qx^{\alpha}) = qE_{\alpha}(qx^{\alpha}),\\
&D_x^{\alpha}\sin_{\alpha}(x^{\alpha}) = \cos_{\alpha}(x^{\alpha}), D_x^{\alpha}\cos_{\alpha}(x^{\alpha}) = -\sin_{\alpha}(x^{\alpha}),
\end{aligned} \quad (7)$$

where λ, μ, C, and q are constants and k is an integer, while $\sin_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty}(-1)^k x^{(2k+1)\alpha}/\Gamma(1+(2k+1)\alpha)$ and $\cos_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty}(-1)^k x^{2k\alpha}/\Gamma(1+2k\alpha)$.

Definition 2 (see [1]). Let function $u(x)\in C_{\alpha}[a,b]$; then, the definition of α-order local fractional integral of $u(x)$ in the integral $[a,b]$ is as follows:

$$\begin{aligned}
{}_aI_b^{\alpha}u(x) &= \frac{1}{\Gamma(1+\alpha)}\int_a^b u(t)(dt)^{\alpha}\\
&= \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta x_k\longrightarrow 0}\sum_{k=0}^{N-1}u(x_k)(\Delta x_k)^{\alpha}, (0<\alpha\leq 1),
\end{aligned} \quad (8)$$

where $\Delta x_k = x_{k+1} - x_k$ with $x_0 = a < x_1 < \cdots < x_{N-1} < x_N = b$.

The local fractional integral has some basic properties [1]:

$$\begin{aligned}
&{}_aI_b^{\alpha}(\lambda f(x)+\mu g(x))=\lambda_a I_b^{\alpha}f(x)+\mu_a I_b^{\alpha}g(x),\\
&{}_aI_b^{\alpha}[(D_x^{\alpha}f(x))g(x)]=f(x)g(x)|_a^b-{}_aI_b^{\alpha}[f(x)(D_x^{\alpha}g(x))],\\
&{}_0I_x^{\alpha}C=\frac{Cx^{\alpha}}{\Gamma(1+\alpha)},\\
&{}_0I_x^{\alpha}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}=\frac{x^{(k+1)\alpha}}{\Gamma(1+(k+1)\alpha)},\\
&{}_0I_x^{\alpha}E_{\alpha}(qx^{\alpha})=\frac{E_{\alpha}(qx^{\alpha})-1}{q},\\
&{}_0I_x^{\alpha}\sin_{\alpha}(x^{\alpha})=1-\cos_{\alpha}(x^{\alpha}),\,{}_0I_x^{\alpha}\cos_{\alpha}(x^{\alpha})=\sin_{\alpha}(x^{\alpha}).
\end{aligned}\tag{9}$$

3. Local Fractional HPM for the Class of Local Fractional IDEs

In this section, we establish the local fractional HPM for the class of IDEs (2). For convenience, we rewrite Equation (2) as follows:

$$\begin{aligned}
A_{\alpha}(u)&=L_{\alpha}(u)+I_{\alpha}(u)=0,\\
L_{\alpha}(u)&=\frac{d^{\alpha}u(x)}{dx^{\alpha}}-f(x),\\
I_{\alpha}(u)&=-\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(x,t)u(t)(dt)^{\alpha},
\end{aligned}\tag{10}$$

where A_{α} represents the local fractional operator.

In view of the local fractional HPM [17], we construct the local fractional homotopy $H_{\alpha}(u,p^{\alpha})$, $u\in R$, and $p\in[0,1]$ by the following:

$$H_{\alpha}(u,p^{\alpha})=(1-p^{\alpha})(L_{\alpha}(u)-L_{\alpha}(u_0))+p^{\alpha}(L_{\alpha}(u)+I_{\alpha}(u))=0,\tag{11}$$

with the embedded parameter p^{α} which monotonously changes from 0 to 1 leads to the result that the easy-to-solve equation $L_{\alpha}(u)-L_{\alpha}(u_0)=0$ continuously deforms to the original equation $L_{\alpha}(u)+I_{\alpha}(u)=0$. Using the constructed homotopy $H_{\alpha}(u,p^{\alpha})$, we can continuously trace a curve which is implicitly defined from a starting point

$$H_{\alpha}(u,0)=L_{\alpha}(u)-L_{\alpha}(u_0)=0,\tag{12}$$

to a solution function

$$H_{\alpha}(u,1)=L_{\alpha}(u)+I_{\alpha}(u)=0.\tag{13}$$

From the perspective of topology, the above changing process is called a deformation. In this deformation, $L_{\alpha}(u)-L_{\alpha}(u_0)$ and $L_{\alpha}(u)+I_{\alpha}(u)$ are homotopic.

Thus, the fractional homotopy $H_{\alpha}(u,p^{\alpha})$ in Equation (11) can be written as below:

$$\begin{aligned}
H_{\alpha}(u,p^{\alpha})&=(1-p^{\alpha})\left[\frac{d^{\alpha}u(x)}{dx^{\alpha}}-f(x)-\left(\frac{d^{\alpha}u_0(x)}{dx^{\alpha}}-f(x)\right)\right]\\
&\quad+p^{\alpha}\left[\frac{d^{\alpha}u(x)}{dx^{\alpha}}-f(x)-\frac{p^{\alpha}}{\Gamma(1+\alpha)}\int_0^1 g(x,t)u(t)(\mathrm{d}t)^{\alpha}\right]\\
&=\frac{\mathrm{d}^{\alpha}u(x)}{\mathrm{d}x^{\alpha}}-\frac{d^{\alpha}u_0(x)}{dx^{\alpha}}+p^{\alpha}\left(\frac{d^{\alpha}u_0(x)}{dx^{\alpha}}-f(x)\right.\\
&\quad\left.-\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(x,t)u(t)(dt)^{\alpha}\right)=0.
\end{aligned}\tag{14}$$

Substituting the series u expanded by the fractional homotopy parameter p^{α}

$$u=v_0^{\alpha}+p^{\alpha}v_1^{\alpha}+p^{2\alpha}v_2^{\alpha}+p^{3\alpha}v_3^{\alpha}+\cdots,\tag{15}$$

into Equation (14) and comparing the coefficients of the same power of p^{α}, we obtain a set of fractional equations:

$$\begin{aligned}
&p^0:\frac{d^{\alpha}v_0^{\alpha}(x)}{dx^{\alpha}}-\frac{d^{\alpha}u_0(x)}{dx^{\alpha}}=0,\\
&p^{\alpha}:\frac{d^{\alpha}v_1^{\alpha}(x)}{dx^{\alpha}}+\frac{d^{\alpha}u_0(x)}{dx^{\alpha}}-f(x)\\
&\quad-\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(x,t)v_0^{\alpha}(t)(dt)^{\alpha}=0,\\
&p^{2\alpha}:\frac{d^{\alpha}v_2^{\alpha}(x)}{dx^{\alpha}}-\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(x,t)v_1^{\alpha}(t)(dt)^{\alpha}=0,\\
&p^{3\alpha}:\frac{d^{\alpha}v_3^{\alpha}(x)}{dx^{\alpha}}-\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(x,t)v_2^{\alpha}(t)(dt)^{\alpha}=0,\ \vdots\ .
\end{aligned}\tag{16}$$

Here u_0 is assumed to be an initial approximate solution of Equation (2). Generally, the initial approximation v_0^{α} or u_0 can be freely chosen. Solving above set of fractional equations, we can obtain solutions v_0^{α}, v_1^{α}, v_2^{α}, v_3^{α}, and so on.

Setting $p^{\alpha}\longrightarrow 1$ and using Equation (15), we finally arrive at an approximate solution of Equation (2).

$$u=\lim_{p^{\alpha}\longrightarrow 1}\sum_{k=0}^{\infty}v_k^{\alpha}(x)=v_0^{\alpha}+v_1^{\alpha}+v_2^{\alpha}+v_3^{\alpha}+\cdots.\tag{17}$$

As pointed by He [18], the series (17) has convergence in most cases and the convergent rate is determined by $L_{\alpha}(u)$ when $\alpha=1$. For the case of convergence, the series (17) can reach a closed form solution.

4. A Concrete Example

In this section, we apply the previously established local fractional HPM to the local fractional IDE (4).

Firstly, we construct such a fractional homotopy:

$$H_\alpha(u,p^\alpha)=\frac{d^\alpha u(x)}{dx^\alpha}-\frac{d^\alpha u_0(x)}{dx^\alpha}+p^\alpha\left[\frac{d^\alpha u_0(x)}{dx^\alpha}-3E_\alpha(3x^\alpha)+\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)x^\alpha-\frac{1}{\Gamma(1+\alpha)}\cdot\int_0^1\frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)}u(t)(dt)^\alpha\right]=0. \tag{18}$$

Secondly, we substitute Equation (15) into Equation (18), and then, the comparison of the coefficients of $p^{j\alpha}(j=0,1,2,\cdots)$ gives a system of fractional equations:

$$p^0:\frac{d^\alpha v_0^\alpha(x)}{dx^\alpha}-\frac{d^\alpha u_0(x)}{dx^\alpha}=0, \tag{19}$$

$$p^\alpha:\frac{d^\alpha v_1^\alpha(x)}{dx^\alpha}+\frac{d^\alpha u_0(x)}{dx^\alpha}-3E_\alpha(3x^\alpha)+\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)x^\alpha-\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)}v_0^\alpha(t)(dt)^\alpha=0, \tag{20}$$

$$p^{2\alpha}:\frac{d^\alpha v_2^\alpha(x)}{dx^\alpha}-\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)}v_1^\alpha(t)(dt)^\alpha=0, \tag{21}$$

$$p^{3\alpha}:\frac{d^\alpha v_3^\alpha(x)}{dx^\alpha}-\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)}v_2^\alpha(t)(dt)^\alpha=0,\vdots. \tag{22}$$

In view of the arbitrariness of v_0^α or u_0, here we set

$$\frac{d^\alpha v_0^\alpha(x)}{dx^\alpha}=\frac{d^\alpha u_0(x)}{dx^\alpha}=3E_\alpha(3x^\alpha)-\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)x^\alpha, \tag{23}$$

and then, from Equations (20)–(23), we have

$$v_0^\alpha=E_\alpha(3x^\alpha)-\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\cdot\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha};\ \frac{d^\alpha v_1^\alpha(x)}{dx^\alpha}-\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\cdot\left(1-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)x^\alpha=0, \tag{24}$$

namely,

$$v_1^\alpha=\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\left(1-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)\cdot\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha};\ \frac{d^\alpha v_2^\alpha(x)}{dx^\alpha}-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\cdot\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\left(1-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)x^\alpha=0, \tag{25}$$

namely,

$$v_2^\alpha=\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\cdot\left(1-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha};\ \frac{d^\alpha v_3^\alpha(x)}{dx^\alpha}-\frac{3^2\Gamma^2(1+3\alpha)}{\Gamma^2(1+2\alpha)\Gamma^2(1+4\alpha)}\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\cdot\left(1-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)x^\alpha=0, \tag{26}$$

namely,

$$v_3^\alpha=\frac{3^2\Gamma^2(1+3\alpha)}{\Gamma^2(1+2\alpha)\Gamma^2(1+4\alpha)}\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\cdot\left(1-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha};\ \frac{d^\alpha v_4^\alpha(x)}{dx^\alpha}-\frac{3^3\Gamma^3(1+3\alpha)}{\Gamma^3(1+2\alpha)\Gamma^3(1+4\alpha)}\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\cdot\left(1-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)x^\alpha=0, \tag{27}$$

namely,

$$v_4^\alpha=\frac{3^3\Gamma^3(1+3\alpha)}{\Gamma^3(1+2\alpha)\Gamma^3(1+4\alpha)}\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)}-\frac{E_\alpha(3)-1}{3}\right)\cdot\left(1-\frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha};\vdots. \tag{28}$$

Finally, we obtain an approximate solution of Equation (4).

$$
\begin{aligned}
u = \lim_{p^{\alpha}\longrightarrow 1}\sum_{k=0}^{\infty} v_k^{\alpha}(x)p^{k\alpha} = E_{\alpha}(3x^{\alpha}) - \left(\frac{E_{\alpha}(3)}{\Gamma(1+\alpha)} - \frac{E_{\alpha}(3)-1}{3}\right)\\
\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha} + \left(1 + \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right.\\
\left. + \frac{3^2\Gamma^2(1+3\alpha)}{\Gamma^2(1+2\alpha)\Gamma^2(1+4\alpha)} + \frac{3^3\Gamma^3(1+3\alpha)}{\Gamma^3(1+2\alpha)\Gamma^3(1+4\alpha)} + \cdots\right)\\
\times\left(\frac{E_{\alpha}(3)}{\Gamma(1+\alpha)} - \frac{E_{\alpha}(3)-1}{3}\right)\left(1 - \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)\\
\cdot\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha},
\end{aligned}
\tag{29}
$$

which can be simplified as follows:

$$
\begin{aligned}
u = E_{\alpha}(3x^{\alpha}) - \lim_{m\longrightarrow\infty}\frac{3^{m-1}\Gamma^{m-1}(1+3\alpha)}{\Gamma^{m-1}(1+2\alpha)\Gamma^{m-1}(1+4\alpha)}\\
\cdot\left(\frac{E_{\alpha}(3)}{\Gamma(1+\alpha)} - \frac{E_{\alpha}(3)-1}{3}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha}.
\end{aligned}
\tag{30}
$$

Obviously, the nth-order approximate solution of Equation (4) has the following form:

$$
\begin{aligned}
u_n = E_{\alpha}(3x^{\alpha}) - \frac{3^n\Gamma^n(1+3\alpha)}{\Gamma^n(1+2\alpha)\Gamma^n(1+4\alpha)}\\
\cdot\left(\frac{E_{\alpha}(3)}{\Gamma(1+\alpha)} - \frac{E_{\alpha}(3)-1}{3}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha}, (n = 0, 1, 2, \cdots).
\end{aligned}
\tag{31}
$$

When the condition

$$
q = \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)} < 1, \tag{32}
$$

holds, the limit of Equation (30) exists. In this case, we obtain

$$
u = E_{\alpha}(3x^{\alpha}). \tag{33}
$$

In Figure 1, we show the curve of the condition q with fractional order α, where the dashed line represents $q = 1$. We can see from Figure 1 that there exists a unique value $\alpha_0 \in (0.7, 0.75)$ such that $q(\alpha_0) = 1$. At the same time, with the help of Mathematica, we have

$$
q(0.73) \approx 1.02522329079942, q(0.74) \approx 0.9916413061494614. \tag{34}
$$

Thus, we can more accurately determine the range of α_0 as

$$
0.73 < \alpha_0 < 0.74. \tag{35}
$$

This tells that $q(\alpha) < q(\alpha_0) = 1$ if and only if $\alpha_0 < \alpha \leq 1$.

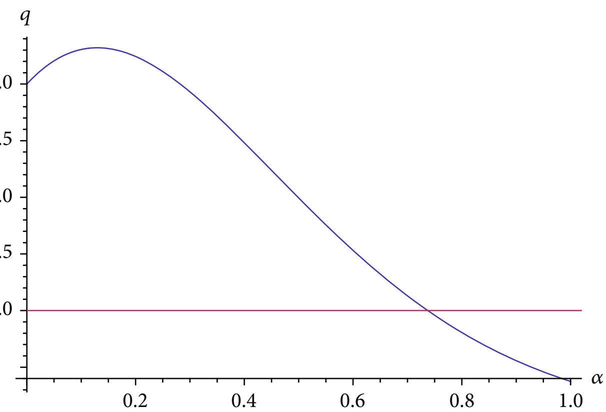

Figure 1: Curve of the condition q with fractional order α.

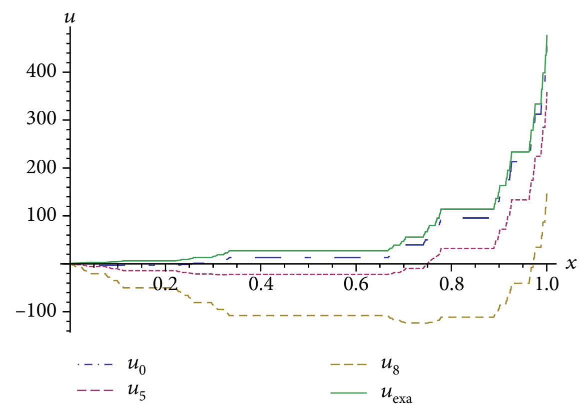

Figure 2: Divergent asymptotic solutions and exact solution in the case $\alpha = \ln 2/\ln 3$.

In Figure 2, the initial solution and 5th-order and 8th-order approximate solutions u_0, u_5, and u_8 and the exact solution u_{exa} are shown by constraining them to a Cantor set with dimension $\alpha = \ln 2/\ln 3 \approx 0.631$ which does not satisfy the convergence condition $\alpha_0 < \alpha \leq 1$. In this case, u_{exa}, u_0, u_5, and u_8 form a sequence of divergent approximate solutions.

Since the initial approximation v_0^{α} or u_0 possesses arbitrariness as mentioned earlier, if we set again

$$
\frac{d^{\alpha}v_0^{\alpha}(x)}{dx^{\alpha}} = \frac{d^{\alpha}u_0(x)}{dx^{\alpha}} = 3E_{\alpha}(3x^{\alpha}), \tag{36}
$$

then the similar process yields

$$
v_0^{\alpha} = u_0 = E_{\alpha}(3x^{\alpha}), \tag{37}
$$

$$
v_1^{\alpha} = v_2^{\alpha} = v_3^{\alpha} = \cdots = 0, \tag{38}
$$

from which we finally reach solution (33). So, for any $0 < \alpha \leq 1$, the solution (33) is always the exact solution of Equation (4). That is to say, if we chose an appropriate initial approximation v_0^{α} or u_0, then the operation can be considerably simplified. More importantly, the convergence condition that the sequence of approximate solutions depends on fractional order can be removed.

It should be noted that when $\alpha = 1$ and $u(x)$, $f(x)$, and $g(x, t)$ are all the continuous and differentiable functions, solution (33) reduces to $u = e^{3x}$ which is the known exact solution of the following IDE [19]:

$$\frac{du(x)}{dx} - 3e^{3x} + \frac{1}{3}\left(2e^3 + 1\right)x - \int_0^1 3xtu(t)dt = 0. \tag{39}$$

5. He-Laplace Method and Comparison

As Deng and Ge [21] pointed out, He-Laplace method has a simple and reliable algorithm and it can be coupled with the HPM or the VIM for solving various nonlinear models, shedding a bright light on fractal calculus. A newest typical example of He-Laplace method to illustrate its simplicity, directness, strength, and great prospects can be found in [22]. In what follows, we employ the local fractional version of He-Laplace method [23] to solve the local fractional IDE (4).

Taking the local fractional Laplace transform on Equation (4), we can gain

$$L\left(\frac{d^\alpha u(x)}{dx^\alpha}\right) = L\left[3E_\alpha(3x^\alpha) + \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)x^\alpha\right] + L\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} u(t)(dt)^\alpha\right], \tag{40}$$

$$s^\alpha L(u(x)) - u(0) = L\left[3E_\alpha(3x^\alpha) + \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)x^\alpha\right] + L\left[\frac{1}{\Gamma(1+\alpha)} \cdot \int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} u(t)(dt)^\alpha\right], \tag{41}$$

$$L(u(x)) = \frac{u(0)}{s^\alpha} + \frac{1}{s^\alpha}L\left[3E_\alpha(3x^\alpha) + \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)x^\alpha\right] + \frac{1}{s^\alpha}L\left[\frac{1}{\Gamma(1+\alpha)} \cdot \int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} u(t)(dt)^\alpha\right]. \tag{42}$$

Then, the inverse local fractional Laplace transform of Equation (42) gives

$$u(x) = E_\alpha(3x^\alpha) - \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^\alpha + L^{-1}\left\{\frac{1}{s^\alpha}L\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} u(t)(dt)^\alpha\right]\right\}, \tag{43}$$

where $u(0) = 1$ has been assumed and L and L^{-1} are the local fractional Laplace transform operator and inverse operator [1], respectively.

Dealing Equation (43) with the local fractional HPM, we introduce

$$\sum_{n=0}^{\infty} p^{n\alpha}u_n(x) = E_\alpha(3x^\alpha) - \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right) \cdot \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^\alpha + p^\alpha L^{-1}\left\{\frac{1}{s^\alpha}L\left[\frac{1}{\Gamma(1+\alpha)} \cdot \int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} p^{n\alpha}u_n(x)(dt)^\alpha\right]\right\}, \tag{44}$$

and compare the coefficients of the same powers of p^α; then, He's polynomials can be obtained:

$$\begin{aligned}
p^0 &: u_0(x) = E_\alpha(3x^\alpha) - \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha},\\
p^\alpha &: u_1(x) = L^{-1}\left\{\frac{1}{s^\alpha}L\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} u_0(x)(dt)^\alpha\right]\right\}\\
&= \left(1 - \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right)\left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right) \cdot \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha},\\
p^{2\alpha} &: u_2(x) = L^{-1}\left\{\frac{1}{s^\alpha}L\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} u_1(x)(dt)^\alpha\right]\right\}\\
&= \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\left(1 - \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right) \cdot \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha},\\
p^{3\alpha} &: u_3(x) = L^{-1}\left\{\frac{1}{s^\alpha}L\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} u_2(x)(dt)^\alpha\right]\right\}\\
&= \frac{3^2\Gamma^2(1+3\alpha)}{\Gamma^2(1+2\alpha)\Gamma^2(1+4\alpha)}\left(1 - \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right) \cdot \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha},\\
p^{4\alpha} &: u_4(x) = L^{-1}\left\{\frac{1}{s^\alpha}L\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{3x^\alpha t^\alpha}{\Gamma(1+\alpha)} u_3(x)(\mathrm{d}t)^\alpha\right]\right\}\\
&= \frac{3^3\Gamma^3(1+3\alpha)}{\Gamma^3(1+2\alpha)\Gamma^3(1+4\alpha)}\left(1 - \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right) \cdot \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha}, \vdots .
\end{aligned} \tag{45}$$

We therefore obtain an approximate solution of Equation (4):

$$u = \lim_{p^\alpha \longrightarrow 1} \sum_{n=0}^{\infty} p^{n\alpha} u_n^\alpha(x) = E_\alpha(3x^\alpha) - \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right) \cdot \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} x^{2\alpha} + \left(1 + \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)} + \frac{3^2\Gamma^2(1+3\alpha)}{\Gamma^2(1+2\alpha)\Gamma^2(1+4\alpha)} + \frac{3^3\Gamma^3(1+3\alpha)}{\Gamma^3(1+2\alpha)\Gamma^3(1+4\alpha)} + \cdots\right) \times \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)\left(1 - \frac{3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\right) \cdot \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} x^{2\alpha}, \tag{46}$$

which is the same as solution (29). It is not difficult to see that solution (46) has the same nth-order approximate solution (31), and when the condition (32) holds, the limit of solution (46) gives the same exact solution (33).

Through comparison, we find that the computational difficulty of solving the local fractional IDE (4) by the above two methods is about the same. When using the local fractional HPM, we need to introduce an appropriate initial approximate solution $v_0(x)$, while the local fractional He-Laplace method uses a known initial value $u(0)$. Generally speaking, it is more difficult to choose an initial approximate solution $v_0(x)$ than to find an initial value $u(0)$. For the latest valuable work on the modified HPM, we suggest to refer to Refs. [24, 25].

6. Conclusion

In summary, we have established the local fractional HPM for the class of local fractional IDEs (1). Based on the established local fractional HPM, an explicit and exact Mittag–Leffler function solution (33) of the local fractional IDE (4) is obtained by selecting two different initial solutions (24) and (37). The comparison shows that the local fractional He-Laplace method [23] can also obtain the same solution (33), but when the initial approximate solution is not easy to choose, the local fractional He-Laplace method [23] has certain advantages over the method used in this paper. The obtained results show that if we choose the approximate solutions appropriately, for example solution (37), then the calculation can be considerably simplified and that the sequence of approximate solutions generated by the local fractional HPM can directly approach the real solution without the influence of fractional order. However, for the selected initial approximate solution (24), we obtained a sequence of approximate solutions converging the real solution (33) in a certain range of the fractional order α, i.e., $\alpha_0 < \alpha \leq 1$. At the same time, there is a divergence interval $(0, \alpha_0)$ which depends on the fractional order α. Here, $q(\alpha_0) = 1$, and an approximate range of α_0 is $0.73 < \alpha_0 < 0.74$. That is to say, α_0 is the critical value of convergence and divergence related to fractional order α of the obtained sequence of approximate solutions. This is different from the HPM for integer-order DEs, all the sequences of approximate solutions of which either converge or diverge, and there is no such a critical value of convergence and divergence. When the nth-order approximate solution (31) is constrained to a Cantor set with dimension $\alpha = \ln 2/\ln 3 \approx 0.631$, this paper shows in Figure 2 a sequence of divergent approximate solutions. This paper fails to describe a sequence of convergent approximate solutions of (31), which is due to the complexity of the numerical simulation of fractal set. How to constrain solution (31) to other fractal sets and show some sequences of convergent approximate solutions? This is a question worth exploring. Besides, the research on qualitative behaviors of Equation (4) and other fractional IDEs is worth discussing. Some recent meaningful results of this research can be found in [26–28]. In 2007, Wang and He [29] took three concrete IDEs as examples to illustrate the effectiveness of the VIM for various IDEs. Based on this fact, we conclude that the local fractional version of VIM can also solve the local fractional IDE (4). In fact, the main steps of the local fractional VIM for Equation (4) can be summarized as follows: (i) identifying Lagrange multiplier $\lambda_\alpha = -1$, (ii) determining the iterative formula of solution:

$$u_{n+1}(x) = u_n(x) - \frac{1}{\Gamma(1+\alpha)} \int_0^x \left(\frac{d^\alpha u_n(\xi)}{d\xi^\alpha} + F(u_n(\xi))\right) \cdot (d\xi)^\alpha, (n = 0, 1, 2 \cdots), \tag{47}$$

with

$$F(u_n(\xi)) = -3E_\alpha(3\xi^\alpha) + \left(\frac{E_\alpha(3)}{\Gamma(1+\alpha)} - \frac{E_\alpha(3)-1}{3}\right)\xi^\alpha - \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{3\xi^\alpha t^\alpha}{\Gamma(1+\alpha)} u_n(t)(dt)^\alpha, \tag{48}$$

and (iii) selecting the initial approximate solution $u_0(x) = E_\alpha(3x^\alpha)$ to obtain the exact solution $u(x) = \lim_{n \longrightarrow \infty} u_n(x) = E_\alpha(3x^\alpha)$ by using the determined iterative formula (47). Nevertheless, it is still worth trying to find an appropriate initial approximate solution and get the approximate solution (29) or (46) by Equation (47).

Data Availability

The data in the manuscript are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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References

[1] X. J. Yang, D. Baleanu, and H. M. Srivastava, *Local Fractional Integral Transforms and Their Applications*, Academic Press, London, 2016.

[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1998.

[3] Y. Zhao, D. F. Cheng, and X. J. Yang, "Approximation solutions for local fractional Schrödinger equation in the one-dimensional Cantorian system," *Advances in Mathematical Physics*, vol. 2013, Article ID 291386, 5 pages, 2013.

[4] A. M. A. El-Sayed, S. Z. Rida, and A. A. M. Arafa, "Exact solutions of fractional-order biological population model," *Communications in Theoretial Physics*, vol. 52, no. 6, pp. 992–996, 2009.

[5] S. Zhang and H. Q. Zhang, "Fractional sub-equation method and its applications to nonlinear fractional PDEs," *Physics Letters A*, vol. 375, no. 7, pp. 1069–1073, 2011.

[6] S. X. Deng and X. X. Ge, "The variational iteration method for Whitham-Broer-Kaup system with local fractional derivatives," *Thermal Science*, vol. 26, no. 3, pp. 2419–2426, 2022.

[7] S. Zhang, B. Cai, and B. Xu, "Variable separation method for nonlinear time fractional biological population model," *International Journal of Numerical Methods for Heat and Fluid Flow*, vol. 25, no. 7, pp. 1531–1541, 2015.

[8] C. Li and F. Zeng, "The finite difference methods for fractional ordinary differential equations," *Numerical Functional Analysis and Optimization*, vol. 34, no. 2, pp. 149–179, 2013.

[9] M. Nadeem, J. H. He, and A. Islam, "The homotopy perturbation method for fractional differential equations: part 1 Mohand transform," *International Journal of Numerical Methods for Heat and Fluid Flow*, vol. 31, no. 11, pp. 3490–3504, 2021.

[10] S. Habib, A. Batool, A. Islam, M. Nadeem, and J. H. He, "Study of nonlinear hirota-satsuma coupled KdV and coupled mKdV system with time fractional derivative," *Fractals*, vol. 29, article 2150108, no. 5, 2021.

[11] Y. Wang, Y. F. Zhang, J. G. Liu, and M. Iqbal, "A short review on analytical methods for fractional equations with He's fractional derivative," *Thermal Science*, vol. 21, no. 4, pp. 1567–1574, 2022.

[12] B. Xu, Y. F. Zhang, and S. Zhang, "Line soliton interactions for shallow ocean-waves and novel solutions with peakon, ring, conical, columnar and lump structures based on fractional KP equation," *Advances in Mathematical Physics*, vol. 2021, Article ID 6664039, 15 pages, 2021.

[13] K. M. Kolwankar and A. D. Gangal, "Fractional differentiability of nowhere differentiable functions and dimensions," *Chaos*, vol. 6, no. 4, pp. 505–513, 1996.

[14] Y. Wang and Q. G. Deng, "Fractal derivative model for tsunami travelling," *Fractals*, vol. 27, no. 2, article 1950017, 2019.

[15] J. Fan and X. M. Shang, "Fractal heat transfer in wool fiber hierarchy," *Heat Transfer Research*, vol. 44, no. 5, pp. 399–407, 2013.

[16] J. Fan, N. Zhu, L. L. Wang, Z. Liu, C. Y. Wang, and Y. Liu, "Influence of hierarchic structure on the moisture permeability of biomimic woven fabricusing fractal derivative method," *Advances in Mathematical Physics*, vol. 2015, Article ID 817437, 4 pages, 2015.

[17] Y. Zhang, C. Cattani, and X. J. Yang, "Local fractional homotopy perturbation method for solving non-homogeneous heat conduction equations in fractal domains," *Entropy*, vol. 17, no. 12, pp. 6753–6764, 2015.

[18] J. H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.

[19] M. Tavassoli Kajani, M. Ghasemi, and E. Babolian, "Comparison between the homotopy perturbation method and the sine-cosine wavelet method for solving linear integro-differential equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 1162–1168, 2007.

[20] G. Jumarie, "On the representation of fractional Brownian motion as an integral with respect to (dt) a," *Applied Mathematics Letters*, vol. 18, no. 7, pp. 739–748, 2005.

[21] S. X. Deng and X. X. Ge, "Approximate analytical solution for Phi-four equation with He's fractal derivative," *Thermal Science*, vol. 25, no. 3 Part B, pp. 2369–2375, 2021.

[22] J. H. He, G. M. Moatimid, and M. H. Zekry, "Forced nonlinear oscillator in a fractal space," *Facta Universitatis-Series: Mechanical Engineering*, vol. 20, no. 1, pp. 001–020, 2022.

[23] H. Y. Zhang, M. Nadeem, A. Rauf, and Z. G. Hui, "A novel approach for the analytical solution of nonlinear time-fractional differential equations," *International Journal of Numerical Methods for Heat and Fluid Flow*, vol. 31, no. 4, pp. 1069–1084, 2021.

[24] X. X. Li and C. H. He, "Homotopy perturbation method coupled with the enhanced perturbation method," *Journal of Low Frequency Noise, Vibration & Active Control*, vol. 38, no. 3-4, pp. 1399–1403, 2019.

[25] C. H. He and Y. O. El-Dib, "A heuristic review on the homotopy perturbation method for non-conservative oscillators," *Journal of Low Frequency Noise, Vibration & Active Control*, vol. 41, no. 2, pp. 572–603, 2022.

[26] M. Bohner, O. Tunç, and C. Tunç, "Qualitative analysis of Caputo fractional integro-differential equations with constant delays," *Computational and Applied Mathematics*, vol. 40, no. 6, 2021.

[27] C. Tunç and O. Tunç, "On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 115, no. 3, article 115, 2021.

[28] C. Tunç, Y. H. Yang, O. Tunç, and J. C. Yao, "New and improved criteria on fundamental properties of solutions of integro—delay differential equations with constant delay," *Mathematics*, vol. 9, no. 24, article 3317, 2021.

[29] S. Q. Wang and J. H. He, "Variational iteration method for solving integro-differential equations," *Physics Letters A*, vol. 367, no. 3, pp. 188–191, 2007.