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Generalized Growth and Approximation of Pseudoanalytic Functions on the Disk

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Abstract. McCoy [20] considered the approximation of pseudoanalytic functions (PAF) on the disk. Pseudoanalytic functions are constructed as complex combination of real - valued analytic solutions to the Stokes-Beltrami System. These solutions include the generalized biaxisymmetric potentials. McCoy obtained some coefficient and Bernstein type growth theorems on the disk. The aim of this paper is to generalize the results of McCoy [20]. Moreover, we study the generalized order and generalized type of PAF in terms of Fourier coefficients occurring in its local expansion and optimal approximation errors in Bernstein sense on the disk. Our approach and method are different from those of McCoy [20].

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1 Introduction

In Bernstein work [7,13,25,27] there is an implicit characterization of the solutions of a system of partial differential equations since the real and imaginary parts of the real analytic function solve the Cauchy-Riemann system. Thus, there is a notion of the harmonic conjugate and its role in the approximation process. However, this concept is not found in the function theory of the generalized biaxisymmetric potentials (GBASP) equation and is necessary for a complete function theoretic generalization

[1]This work was done in the memory of Prof. H.S.Kasana, Senior Associate, ICTP, Trieste, Italy

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of Bernstein type. GBASP that are harmonic at the origin may be expanded, in analogy with the Taylor's series for analytic functions of a single complex variable, in a convergent series of homogeneous harmonic polynomials on an open set.

Definition of pseudoanalytic functions. The class A_D of functions $w(z)$ which are analytic in a fixed domain D can be characterized by four properties (1). A_D is a real vector space (2). If $w(z)$ belongs to A_D and $w(z_o) = 0$ for some z_o in D, then the quotient $\frac{w(z)}{(z-z_0)}$ is continuous at z_0. (3). The 1 and i belong to A_D. (4). A_D is maximal with respect to properties (1),(2),(3).

In fact the first three properties imply that the existence of the complex derivative $w^{'}(z)$ at every point of D is necessary in order that a function w should belong to A_D, and property (4) implies that this condition is sufficient.

The theory of analytic functions of a complex variable occupies a central place in analysis and it is not surprising that mathematical literature abounds in generalizations. In some generalizations one extends the domain of the functions considered, or their range, or both.If we restrict ourselves to functions from Riemann surfaces to Riemann surfaces, we encounter two well known and very useful generalizations of analytic functions: interior functions and quasi-conformal functions. The class of quasi-conformal functions contains all analytic functions as well as many others. For this reason the theory of quasi-conformal functions can not have the inner rigidity and harmony of classical function theory. For example, quasi-conformal functions do not have the unique continuation property which Riemann considered to be the most characteristic feature of analytic functions. Pseudoanalytic functions, on the other hand, do possess the unique continuation property, and each class of pseudoanalytic functions has almost as much structure as the class of analytic functions. In particular, the operation of complex differentiation and complex integration have meaningful counterparts in the theory of pseudoanalytic functions and this theory generalizes not only the Cauchy-Riemann approach to function theory but also that of Weierstrass. Due to above useful properties of pseudoanalytic functions, it is reasonable to study the growth properties of these functions.

Pseudoanalytic functions are roughly speaking, solutions of generalized Cauchy - Riemann equations. Such functions were considered by Picard and Beltrami, but the first significant result was obtained by Carleman in 1933. The theory of pseudoanalytic functions was developed from the point of view of partial differential equations, much of the motivation being provided by problems in Mechanics of continua. Analogues of complex differentiation and integration for an elliptic partial differential were used by Beltrami [4] in the special case of axially symmetric potential. In a systematic way these concepts, as well as formal powers and power series, have been introduced in 1943 by Gilbert and the Bers [9,10,11] for equations of special forms (cf.6). The original version of a general theory of pseudoanalytic functions was influenced by the work of *Položii* [21,22,23,24] and made essential use of Carleman's unique continuation theorem. This theory applied to equations with *Hölder* continuously differentiable coefficients.

Pseudoanalytic functions are constructed as complete combinations of real - valued function pairs that are analytic solutions of Stokes - Beltrami System (SBS); a generalization of the Cauchy - Riemann equations that is linked to the GBASP equation by eliminating one of the dependent variables from the system. Stokes- Beltrami systems have their origin in the study of axially symmetric solutions of the n- dimensional Laplace equation

$$\nabla_n^2 f = 0, \tag{1.1}$$

where

$$\nabla_n^2 = \Sigma_{i=1}^n \frac{\partial^2}{\partial i^2}. \tag{1.2}$$

Thus, if we consider axially symmetric solutions of (1.1), namely, in those which depend on two variables

$$x = x^1, y = [\Sigma_{i=2}^n (x^i)^2]^{1/2}, \tag{1.3}$$

the x-axis being the axis of symmetry. Under these restrictions f becomes a function $\phi(x, y)$ of the two variables x and y, and satisfies the equation

$$\frac{\partial}{\partial x}(y^p \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(y^p \frac{\partial \phi}{\partial y}) = 0. \tag{1.4}$$

where $p = n - 2$ and $y \geq 0$. Accordingly, ϕ may be regarded as an axially symmetric potential in a space of $n = p + 2$ dimensions. Here, it will prove convenient to study the generalized axially symmetric potential equation (1.4) in the context of the generalized Stokes-Beltrami system . In the case $p = 0$, the Cauchy-Riemann system is retrived, while the case $p = 1$ was the subject of a series of published circa 1880. There, Beltrami founded a generalized theory of analytic functions later to be developed in a systematic manner by Bers and Gilbert and Weinstein.

Pseudoanalytic functions provide sufficient basis for the transformation of Bernstein's ideas through transform and special function methods.The real part of pseudoanalytic function i.e., eliminating the harmonic conjugate gives the theory of GBASP. The GBASP equation frequently found in the summability theory of [3] Jacboi series as

$$r\frac{\partial}{\partial r}\left\{r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta) r\frac{\partial u}{\partial r}\right\} + \frac{\partial}{\partial \theta}\left\{r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)\frac{\partial u}{\partial \theta}\right\} = 0$$

$$\rho^{(\alpha,\beta)}(\theta) = (\sin\theta/2)^{2\alpha+1}(\cos\theta/2)^{2\beta+1}, \alpha \geq \beta > \frac{-1}{2}$$

where (r, θ) are the plane polar coordinates.The domain of the potential u is a simply connected region with smooth boundary in the upper half plane $C^+ = C \cap Re(z) \geq 0$. The existence of a harmonic conjugate ν of u is implied in the sense of the generalized Stokes - Beltrami System (SBS);

$$\begin{aligned} r\frac{\partial \nu}{\partial r} &= -r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)\frac{\partial u}{\partial \theta} \\ \frac{\partial \nu}{\partial \theta} &= r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta) r\frac{\partial u}{\partial r} \end{aligned}$$

a system that reduces to the Cauchy-Riemann equations of analytic function theory in the limit $\alpha = \beta = -1/2$. Following along the lines of analytic function theory, a pseudoanalytic function [2, 12,29] (PAF) is defined as the complex combination

$$F(re^{i\theta}) = u(r, \theta) + i\nu(r, \theta)$$

of a real-valued analytic function pair formed from the potential u and the principal branch of its harmonic conjugate $\nu = \nu(r, \theta)$. In particular, a PAF whose domain Ω^+ intersects the x-axis in a single component may be analytically continued across the axis by reflextion to Ω^- as an even function on $\Omega : \Omega^+ \cup \Omega^-$ provided that $\frac{\partial u(r,0+\pi k)}{\partial \theta} = \frac{\partial \nu(r,0+\pi k)}{\partial \theta} = 0, 1$[12].

The Bernstein theorem identify a real analytic function on the closed unit disk as the restriction of an analytic function defined on an open disk of radius $R > 1$ by computing R from the sequence of minimal errors generated from optimal polynomials approximates. Here we use this concept for pseudoanalytic functions (PAF). The disk D_R of maximum radius on which a PAF F exists, is designated by $F \in P(D_R)$. If F is an entire PAF, it has no singularities in the finite C^+-plane and writes $F \in P(C)$.

The maximum modulus of a pseudoanalytic function $F \in P(D_R)$ is defined on the closure of a disk D_R as

$$M_r(F) = \max\{|F(re^{i\theta})| : re^{i\theta} \in \overline{D}_R\}, r < R.$$

Let p and q be two positive, strictly increasing to infinity differentiable functions $]0 + \infty]$ to $]0, +\infty[$ such that for every $c > 0$ satisfies:

$$\begin{aligned} \lim_{x\to\infty} \frac{p(cx)}{p(x)} &= 1, \\ \lim_{x\to\infty} \frac{q(1 + xw(x))}{q(x)} &= 1, \lim_{x\to\infty} w(x) = 0, \\ \lim_{x\to\infty} \frac{d(q^{-1}(cp(x)))}{d(\log x)} &\leq b. \end{aligned}$$

$$p\left(\frac{x}{q^{-1}(cp(x))}\right) = (1 + o(x))p(x), \quad \texttt{for } x \to \infty,$$

where $d(u)$ means differential of u.

The $(p, q)-$order and $(p, q)-$type of an entire PAF $F \in P(C)$ (or generalized order and generalized type) are defined respectively by:

$$\rho(p, q) = \limsup_{r\to\infty} \frac{p(\log M(r, f))}{q(\log r)}$$

and

$$\sigma(p, q) = \limsup_{r\to\infty} \frac{p(M(r, f))}{[q(r)]^{\rho(p,q)}}$$

where

$$M(r, f) = \sup_{z\in\overline{D}_R} |f(z)|.$$

Now applying the concept introduced by G.R. MacLane [16] to the measures of order and type for an analytic function on a disk $|z| < R$ by normalizing these definitions relative to the boundary under the transformation $r \to R/(R-r)$. In this direction, a pseudoanalytic function $F \in P(D_R)$ with radial limits is said to be of generalized regular growth $(\rho_0(p,q), \sigma_0(p,q))$ if it satisfies

$$\begin{aligned} \rho_0(p,q) &= \limsup_{r\to R} \frac{p(\log M_r(F))}{q(R/(R-r))}, \\ \sigma_0(p,q) &= \limsup_{r\to R} \frac{p(\log M_r(F))}{[q(R/(R-r))]^{\rho_0(p,q)}}, 1 < r < R, \end{aligned}$$

in which case $\rho_0(p,q)$ is referred to as the $(p,q)-$ order of F provided that $0 < \rho_0(p,q) < \infty$ and $\sigma_0(p,q)$ the $(p,q)-$type. Normalized to the closed unit disk $\overline{D}_1$ the $M_r(F)$ is a special case of the norm, $\|.\|$,

$$\|F\| = \begin{cases} \|F\|_\delta = \left[\int\int_{\overline{D}_1} |F|^\delta r dr d\phi\right]^{1/\delta}, & 1 \le \delta < \infty \\ \|F\|_\infty = M_1(F), & \delta = \infty. \end{cases}$$

The approximating pseudoanalytic polynomials of (fixed) degree n are taken from the sets

$$\pi_n = \left\{ P : P(re^{i\theta}) = \sum_{k=0}^{n} c_k w_k F_k(r, e^{i\theta}), c_k \text{ real} \right\}, n = 0, 1, 2, \ldots.$$

The optimal approximates minimize the error $\|F - P\|$ for $P \in \pi_n$ in Bernstein sense as

$$E_n(F) = \inf \{\|F - P\| : P \in \pi_n\}, n = 0, 1, 2, \ldots. \tag{1.5}$$

P.A.McCoy [17-19] studied the function - theoretic methods based on the integral operator approach with Bernstein's ideas in the case of weakening the norms measuring the best approximates, and through less stringent analyticity requirements such as replacing the restriction of an entire function on an open disk simply with analyticity to GBASP. Kumar [15] obtained the relationship between the pseudoanalytic functions and Bergmann Gilbert type integral operators for GBASP and polynomial approximation. Bergman [5] and Gilbert [12] generalize the operation of taking the real part. They obtained bounded linear operators which transform analytic functions to solution u, where integral operators are developed to provide the transformation from analytic functions to solutions of corresponding elliptic equation. Bers [8] and Vekua [28] have also extended function theory so that solutions u of elliptic equations can be obtained as $u = Re(F)$, where f is a pseudoanalytic function sharing many of the properties associated with classical analytic functions of single complex variable.

McCoy [20] considered the approximation of pseudoanalytic functions on the disk and obtained some coefficient and Bernstein type growth theorems. Also Kapoor and Nautiyal [14] characterized the order and type of GBASP's (not necessarily entire) in terms of rates of decay of approximation errors on both sup norm and L^p-norm, $1 \le p < \infty$. It has been noticed that all these authors have not studied the generalized growth of pseudoanalytic functions on the disk. In this paper we study the generalized order and generalized type of PAF in terms of Fourier coefficients occurring in its local expansion and optimal approximation errors in Bernstein sense on the disk. Our results and methods in the present paper are different from all those of authors mentioned above.

2 Generalized Order and Generalized Type with Fourier Coefficients of Pseudoanalytic Functions

Let the pseudoanalytic function $F \in P(D_R)$. In a neighborhood of the origin, a PAF has the local expansion

$$F(re^{i\theta}) = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta}), re^{i\theta} \in D_R,$$

$F_n(re^{i\theta}) = u_n(r,\theta) + i\nu_n(r,\theta), n = 0, 1, 2, \dots,$ and a_n real-valued. Also we have the estimate $|w_n F_n(re^{i\theta})| \sim n^{2\alpha+1} r^{n+\alpha+\beta+1}$ for large n.Put

$$\limsup_{n\to\infty} \frac{p(n)}{q\left[\frac{n}{\log(n^{2\alpha+1}|a_n|R^n)}\right]} = \mu(p,q). \tag{2.1}$$

Lemma 2.1. For every $r > 1$ and $\mu > 0$, the maximum of the function

$$x \to w(x,r) = x\log(r/R) + \frac{x}{q^{-1}(p(x)/\mu)}$$

is reached for $x = x_r$ solution of the equation

$$x = p^{-1}\left\{\mu q\left[\frac{1 - d\log(q^{-1}(p(x)/\mu))/d(\log x)}{\log(R/r)}\right]\right\}.$$

Proof. Let $G(x,\mu) = q^{-1}(p(x)/\mu)$, then

$$w(x,r) = x\log(r/R) + \frac{x}{G(x,\mu)}.$$

The maximum of the function $x \to w(x,r)$ is reached for $x = x_r$, solution of the equation of $\frac{dw(x,r)}{dx} = 0$.

We have

$$\frac{dw(x,r)}{dx} = 0 \Leftrightarrow \log(r/R) + \frac{G(x,\mu) - x\frac{dG(x,\mu)}{dx}}{(G(x,\mu))^2} = 0$$

or

$$G(x,\mu) = \frac{1 - \frac{x}{G(x,\mu)}}{.}\frac{dG(x,\mu)/dx}{\log(R/r)}.$$

Since

$$\frac{dG(x,\mu)}{dx} = \frac{dG(x,\mu)}{d(\log x)}.\frac{d(\log x)}{dx} = \frac{1}{x}\frac{dG(x,\mu)}{d(\log x)},$$

we get

$$G(x,\mu) = \frac{1 - \frac{1}{G(x,\mu)}\frac{dG(x,\mu)}{d(\log x)}}{\log(R/r)} = \frac{1 - \frac{d\log G(x,\mu)}{d(\log x)}}{\log(R/r)},$$

we obtain

$$x = x_r = p^{-1}\left\{\mu q\left[\frac{1 - d\log(q^{-1}(p(x)/\mu))/d(\log x)}{\log(R/r)}\right]\right\}. \tag{2.2}$$

Lemma 2.2. Let $F(re^{i\theta}) = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta}), F \in P(D_R)$. For every $1 < r < R$, we put

$$\begin{aligned} \overline{M}(r,F) &= \sup_{n\in N}\left\{\|a_n w_n\|.r^n, r > 0\right\}, \\ \overline{\rho}_0(p,q) &= \limsup_{r\to R}\frac{p(\log\overline{M}(r,F))}{q(R/(R-r))}, \end{aligned}$$

then

$$\overline{\rho}_0(p,q) \le \mu(p,q) \text{ and } \rho_0(p,q) \le \overline{\rho}_0(p,q).$$

Proof. From (2.1) we have for r sufficiently close to R and $\overline{\mu} = \mu + \varepsilon$,

$$\log(n^{2\alpha+1}|a_n|R^n) \le \frac{n}{q^{-1}\left(\frac{p(n)}{\overline{\mu}}\right)}$$

or

$$\log(n^{2\alpha+1}|a_n|R^n) \le n\log(r/R) + \frac{n}{q^{-1}\left(\frac{p(n)}{\overline{\mu}}\right)}.$$

Using the result

$$\lim_{x\to\infty}\left|\frac{d(q^{-1}(cp(x)))}{d(\log x)}\right| \le b, \log(1+t) = (1+o(t)).t, t \to 0,$$

we get from (2.2)

$$x_r = (1+o(1))p^{-1}(\mu q(R/(R-r))) \text{ as } r \to R.$$

It gives

$$\log(n^{2\alpha+1}|a_n|r^n) \le C_0 p^{-1}(\mu q(R/(R-r))),$$

or

$$\log\overline{M}(r,F) \le C_0 p^{-1}(\mu q(R/(R-r))),$$

using the property of p, we obtain

$$\frac{p(\log\overline{M}(r,F))}{q(R/(R-r))} \le \overline{\mu}.$$

Proceeding the limit supremum as $r \to R$, we get

$$\overline{\rho}_0(p,q) \le \mu. \tag{2.3}$$

Now consider

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n \\ M(r,f) &\le \sum_{n=0}^{\infty}|a_n|r^n \le \sum_{n=0}^{\infty}|a_n|w_n r^n, \end{aligned}$$

substituting $r = \sqrt{r.R}.\sqrt{(r/R)}$ in above we have

$$M(r,f) \leq \sum_{n=0}^{\infty} |a_n| w_n \left(\sqrt{r.R}\right)^n \left(\sqrt{(r/R)}\right)^n, (r/R) < 1,$$

or

$$M(r,f) \leq \sum_{n=0}^{\infty} \sup\left(|a_n| w_n \left(\sqrt{r.R}\right)^n\right) . \left(\sqrt{(r/R)}\right)^n$$

or

$$\begin{aligned} M(r,f) \leq \quad & \overline{M}(r', F) \textstyle\sum_{n=0}^{\infty} \left(\sqrt{(r/R)}\right)^n \\ \leq \quad & \overline{M}(r', F) \left(\frac{1}{1 - \sqrt{(r/R)}}\right), r' = \sqrt{(r.R)} \end{aligned}$$

It gives

$$\log M(r,f) \leq \log \overline{M}(r', f) - \log\left(1 - \sqrt{(r/R)}\right)$$

or

$$\frac{p(\log M(r,f))}{q(R/(R-r))} \leq \frac{p(\log \overline{M}(r', F) - \log(1 - \sqrt{(r/R)}))}{q(R/(R - \sqrt{(r.R)}))} . \frac{q(R/(R - \sqrt{(r.R)}))}{q(R/(R-r))}.$$

Passing to limit we get

$$\rho_0(p,q) \leq \overline{\rho}_0(p,q). \tag{2.4}$$

From (2.3) and (2.4) we get $\rho_0(p,q) \leq \mu(p,q)$.

Theorem 2.1. Let $F(re^{i\theta}) = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta}), F \in P(D_R), re^{i\theta} \in D_R$ such that

$$\limsup_{n\to\infty} \frac{p(n)}{q\left[\frac{n}{\log(|a_n| n^{2\alpha+1} R^n)}\right]} = \mu(p,q) < \infty.$$

Then F is the restriction of pseudoanalytic function in $P(D_R)(R > 1)$ and its $(p,q)-$order $\rho(p,q) = \mu(p,q)$.

Proof. To study the expansion of a PAF in terms of a complete set of particular solutions of the SBS. Let us consider the normalized Jacobi polynomials $R_n(\cos\theta) = P_n^{(\alpha,\beta)}(\cos\theta)/P_n^{(\alpha,\beta)}(1), n = 0, 1, 2, \ldots,$ [1, 17] which form an orthogonal set

$$\begin{aligned} &\int_0^{\pi} R_n(\cos\phi) R_m(\cos\phi) \rho^{(\alpha,\beta)}(\phi) d\phi = w_n^{-1} \delta_{nm} \\ &w_n = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)\Gamma(\beta+1)} \sim O(n^{2\alpha}) \end{aligned}$$

relative to the measure $\rho^{(\alpha,\beta)}(\phi) d\phi$ on $[0,\pi]$. The Abel - Poisson and conjugate Abel-Poisson means [3,26] of these polynomials

$$\begin{aligned} u_n(r,\theta) &= A_r(R_n(\cos\theta)) = r^n R_n(\cos\theta) \\ \nu_n(r,\theta) &= \widetilde{A}_r(R_n(\cos\theta)) \end{aligned}$$

$$= nr^{n+\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)(\cos\theta/2)(\sin\theta/2)/(\alpha+1). \tag{2.5}$$

$(R_{-1}^{(\alpha+1,\beta+1)}(\cos\theta) \equiv 1)$ produce a set of particular solutions [3] that is complete [26] relative to uniform convergence on compacta of a sufficiently small region. In view of the local expansion of PAF and the estimate

$$|w_n F_n(re^{i\theta})| \sim n^{2\alpha}(1 + nr^{\alpha+\beta+1})r^n$$

with the bound $|R_n(\cos\theta)| \leq 1$, a standard argument shows that the Fourier coefficients support the natural domain since a necessary and sufficient condition for $F \in P(D_R)$ is that

$$\limsup_{n\to\infty} |a_n|^{1/n} = R^{-1}.$$

So, for every $1 < r < R$ the series $\sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$ is convergent in D_R when $\sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$ is analytic in D_R.

Now we shall prove that $\mu(p, q)$ is the $(p, q)-$order of F.

By Lemma 2.2, to complete the proof of the theorem it suffices to show that $\rho(p, q) \geq \mu(p, q)$.

By definition of $\rho(p, q)$, we have for every $\varepsilon > 0$ there exists $1 < r_\varepsilon < R$ such that for every $r_\varepsilon < r < R$

$$\log M_r(F) \leq p^{-1}[(\rho(p, q) + \varepsilon).q(R/(R - r))]. \tag{2.6}$$

Let $\overline{D}_{r_0}$ be the closure of a disk D_{r_0} in the natural domain of D_R of a PAF. Such a PAF has the representation [3]

$$F(re^{i\theta}) = \{A_r(.) + i\widetilde{A}_r(.)\}f(r_0 e^{i\theta}), re^{i\theta} \in D_{r_0} \tag{2.7}$$

where the Abel means, as in (2.5), are

$$\begin{aligned} A_r(f(r_0 e^{i\theta}) &= \sum_{n=0}^{\infty} a_n w_n u_n(r, \theta), \\ \widetilde{A}_r(f(r_0 e^{i\theta}) &= \sum_{n=0}^{\infty} a_n w_n \nu_n(r, \theta). \end{aligned}$$

At the boundary (2.7) reduces to

$$F(r_0 e^{i\theta}) = f(r_0 e^{i\theta}) + ir_0^{(\alpha+\beta+1)}\rho^{(\alpha,\beta)}(\theta)\widetilde{f}(r_0 e^{i\theta})$$

where the function f and its conjugate $\widetilde{f}$ are real-valued. The Fourier coefficients are reclaimed by orthogonality as

$$a_n r^n = \int_0^\pi f(re^{i\phi})R_n(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi, r < r_0.$$

Integrating this identity, and applying Holder's inequality we get

$$\begin{aligned} |a_n/(n+2)| &= |\int_0^1 a_n r^n r dr| \leq \int_0^1 \int_0^\pi |f(re^{i\phi})|\rho^{(\alpha,\beta)}(\phi) r dr d\phi \\ &\leq \int_0^1 \int_0^\pi |F(re^{i\phi})|\rho^{(\alpha,\beta)}(\phi) r dr d\phi \\ &\leq \|F\|_\delta.A, A = \|\rho^{(\alpha,\beta)}\|_{\delta'}, \frac{1}{\delta} + \frac{1}{\delta'} = 1, \end{aligned}$$

or

$$|a_n| \leq (n+2)A\|F\|_\delta. \tag{2.8}$$

Now using (2.5) for $1 < r$ sufficiently close to R in above inequality we obtain

$$\log(|a_n|n^{2\alpha+1}R^n) \leq -n\log(r/R) + \log((n+2)n^{2\alpha+1}A) + p^{-1}[(\rho(p,q)+\varepsilon)q(R/(R-r))].$$

The minimum value of right hand side satisfying above inequality is attained at

$$r = R\left\{1 - \frac{1}{q^{-1}\left(\frac{1}{(\rho+\varepsilon)}p\left(\frac{n}{q^{-1}(p(n)/(\rho+\varepsilon))}\right)\right)}\right\}, \rho = \rho(p,q),$$

applying the properties of the function p and q

$$p\left(\frac{x}{q^{-1}(cp(x))}\right) \simeq (1+o(x))p(x) \text{ for } c > 0, x \to \infty,$$

and the properties of logarithm, we get

$$\log(|a_n|n^{2\alpha+1}R^n) \leq c_1 \frac{n}{q^{-1}(p(n)/(\rho+\varepsilon))}$$

where c_1 is a constant. Hence

$$q\left(\frac{c_1 n}{\log(|a_n|n^{2\alpha+1}R^n)}\right) \geq p(n)/(\rho+\varepsilon).$$

Now proceeding to limit supremum as $n \to \infty$ we get

$$\mu(p,q) \leq \rho(p,q).$$

Hence the proof is complete.

Remark 2.1. For $p(x) = q(x) = \log x$, Theorem 2.1 gives part(i) of [20, Thm. 2].
Let $F = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$ be pseudoanalytic function of $(p,q)-$order $\rho = \rho(p,q)$, and write

$$T(p,q) = \limsup_{n\to\infty} \frac{p(n)}{\left\{q\left(\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho(p,q)}}.$$

Now we prove

Lemma 2.3. Let $F = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$. For every $1 < r < R$,

$$\overline{\sigma}_1(p,q) = \limsup_{r\to R} \frac{p(\log \overline{M}(r,F))}{(q(R/(R-r)))^{\rho(p,q)}}$$

then

$$\sigma(p,q) \leq \overline{\sigma}_1(p,q).$$

Proof. Using the same reasoning as in the proof of Lemma 2.2. we get

$$\frac{p(\log M(r,f))}{[q(R/(R-r))]^{\rho(p,q)}} \leq \frac{p(\log(\overline{M}(\sqrt{rR},F) - \log(1-\sqrt{(r/R)})))}{[q(R/(R-\sqrt{rR}))]^{\rho(p,q)}} . \frac{[q(R/(R-\sqrt{rR}))]^{\rho(p,q)}}{[q(R/(R-r))]^{\rho(p,q)}}.$$

Proceeding to the limit supremum as $r \to R$, we get

$$\sigma(p,q) \leq \overline{\sigma}_1(p,q).$$

Theorem 2.2. Let $F = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$ be a pseudoanalytic function on the closed unit disk. If F is of finite generalized $(p,q)-$order $\rho(p,q)$, and

$$T(p,q) = \limsup_{n\to\infty} \frac{p(n)}{\left[q\left(\frac{n}{\log((a_n)n^{2\alpha+1}R^n)}\right)\right]^{\rho(p,q)}} < \infty.$$

Then F is the restriction of a pseudoanalytic function in $P(D_R)(R > 1)$ and its $(p,q)-$type $\sigma(p,q) = T(p,q)$.

Proof. The function F is the restriction of a pseudoanalytic function in $P(D_R)$ by using the arguments in Theorem 2.1. Now first we show that $\sigma(p,q) \leq T(p,q)$.

If $T \equiv T(p,q) < \infty$, by definition of T, for every $\varepsilon > 0$, there exists $n \geq n_\varepsilon$

$$p(n) \leq (T+\varepsilon)\left[q\left(\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right]^{\rho}$$

or

$$\log(|a_n|n^{2\alpha+1}R^n) \leq \frac{n}{q^{-1}\left(\left(\frac{p(n)}{\overline{T}}\right)^{1/\rho}\right)}, \overline{T} = T+\varepsilon, \tag{2.9}$$

for every $n \geq n_\varepsilon$.

Since

$$\log(|a_n|n^{2\alpha+1}r^n) \leq n\log(r/R) + \log(|a_n|n^{2\alpha+1}R^n).$$

By the relation (2.9), we get

$$\log(|a_n|n^{2\alpha+1}r^n) \leq n\log(r/R) + \frac{n}{q^{-1}\left(\left(\frac{p(n)}{\overline{T}}\right)^{1/\rho}\right)}. \tag{2.10}$$

For every $1 < r < R$ and r sufficiently close to R, we put

$$\varphi(x,r) = x\log(r/R) + \frac{x}{q^{-1}\left(\left(\frac{p(x)}{\overline{T}}\right)^{1/\rho}\right)}.$$

If we put $S = S(x,\overline{T},1/\rho) = q^{-1}\left(\left(\frac{p(x)}{\overline{T}}\right)^{1/\rho}\right)$ then

$$\varphi(x,r) = x\log(r/R) + \frac{x}{S},$$

and the maximum of the function $x \to \varphi(x,r)$ is reached for $x = x_r$ solution of the equation

$$\frac{d\varphi(x,r)}{dx} = \frac{\partial\varphi(x,r)}{\partial x} = \log(r/R) + \frac{d}{dx}\left(\frac{x}{S}\right) = 0.$$

or

$$\frac{d\varphi(x,r)}{dx} = 0 \Leftrightarrow \log(r/R) + \frac{S - x\frac{dS}{dx}}{S^2} = 0$$

or

$$S = \frac{1 - \frac{x}{S}\frac{dS}{dx}}{\log(R/r)}.$$

Since

$$\frac{dS}{dx} = \frac{dS}{d\log x}.\frac{d\log x}{dx} = \frac{1}{x}\frac{dS}{d\log x},$$

we put

$$S = \frac{1 - \frac{1}{S}\frac{dS}{d\log x}}{\log(R/r)} = \frac{1 - \frac{d\log S}{d\log x}}{\log(R/r)},$$

or

$$q^{-1}\left(\left(\frac{p(x)}{\overline{T}}\right)^{1/\rho}\right) = \frac{1 - \frac{d\log q^{-1}\left(\left(\frac{p(x)}{\overline{T}}\right)^{1/\rho}\right)}{d\log x}}{\log(R/r)}.$$

It gives

$$x = x_r = p^{-1}\left\{\left[\overline{T}q\left(\frac{1 - d\log\left(q^{-1}\left(\frac{p(x)}{\overline{T}}\right)^{1/\rho}\right)/d(\log x)}{\log(R/r)}\right)\right]^{\rho}\right\}.$$

We have

$$\log(r/R) = \log\left(\frac{r-R}{R} + 1\right) \sim \frac{r-R}{R}\left(\texttt{ as } \frac{r-R}{R} \to 0\right),$$

and

$$\left|\frac{d\left[\log\left(q^{-1}\left(\frac{p(x)}{\overline{T}}\right)^{1/\rho}\right)\right]}{d(\log x)}\right| \leq b,$$

where b is a positive constant. Now we have

$$x_r = (1 + o(1))\rho p^{-1}\left(\overline{T}q(R/(R-r))^{\rho}\right).$$

Using the relation (2.10), we get

$$\log(|a_n|n^{2\alpha+1}r^n) \leq \sup\varphi(x,r)$$

or

$$\log(|a_n|n^{2\alpha+1}r^n) \leq \frac{(1+o(1))p^{-1}\left(\overline{T}(q(R/(R-r)))^{\rho}\right)}{R/(R-r)}.$$

Since $\frac{R}{R-r} > 1$ and $\frac{\rho-1}{\rho} < 1$, then

$$\log(|a_n|n^{2\alpha+1}r^n) \leq Cp^{-1}(\overline{T}(q(R/(R-r)))^{\rho}).$$

Then

$$\log \overline{M}(r,F) \leq Cp^{-1}(\overline{T}(q(R/(R-r)))^{\rho}).$$

Thus

$$\frac{p(\log \overline{M}(r,F))}{(q(R/(R-r)))^{\rho}} \leq \overline{T}.$$

Proceeding to limit supremum for $r \to R$, we get

$$\overline{\sigma}_1(p,q) = \limsup_{r\to R} \frac{p(\log \overline{M}(r,F))}{(q(R/(R-r)))^{\rho}} \leq T.$$

Now using Lemma 2.3, we obtain

$$\sigma(p,q) \leq T(p,q).$$

The result is obviously holds for $T=\infty$. Now we shall show that $\sigma(p,q) \geq T(p,q)$.

Suppose that $\sigma(p,q) < \infty$. By definition of $\sigma(p,q)$, we have for every $\varepsilon > 0$, there exist $r_\varepsilon \in]1, R[$, such that for every $r > r_\varepsilon (R > r > r_\varepsilon > 1)$

$$\log M(r,F) \leq p^{-1}\left[\overline{\sigma}(q(R/(R-r)))^{\rho}\right].$$

Now using (2.8) for $1 < r$ sufficiently close to R with above inequality we get

$$\log(|a_n|n^{2\alpha+1}R^n) \leq -n\log(r/R) + \log((n+2)n^{2\alpha+1}A) + p^{-1}\left[\overline{\sigma}(q(R/(R-r)))^{\rho}\right]$$

or

$$\frac{\log(|a_n|n^{2\alpha+1}R^n)}{n} \leq \eta(r,n)$$

where

$$\eta(r,n) = -\log(r/R) + \frac{1}{n}\log((n+2)n^{2\alpha+1}A) + \frac{1}{n}p^{-1}\left[\overline{\sigma}(q(R/(R-r)))^{\rho}\right].$$

Since

$$\lim_{n\to\infty} \frac{1}{n}\log((n+2)n^{2\alpha+1}A) + \frac{1}{n}p^{-1}\left[\overline{\sigma}(q(R/(R-r)))^{\rho}\right] = 0,$$

we get, for r sufficiently close to R

$$\lim_{n\to\infty} \eta(r,n) = -\log(r/R) = \log(R/r).$$

So we have

$$\eta(r,n) = (1+o(1))\log(R/r), n \to \infty,$$

then

$$\frac{1}{n}\log(|a_n|n^{2\alpha+1}R^n) \leq (1+o(1))\log(R/r). \tag{2.11}$$

Choose

$$r = R\frac{q^{-1}\left(\frac{p(n)}{\overline{\sigma}}\right)^{1/\rho}}{1+q^{-1}\left(\frac{p(n)}{\overline{\sigma}}\right)^{1/\rho}}.$$

Using the relation (2.11) and the properties of the function $t \to \log(t)$, we get for r sufficiently close to R

$$\frac{\log(|a_n|n^{2\alpha+1}R^n)}{n} \leq (1+o(1))((R/r)-1).$$

$$\left[\log(R/r) = \log\left(\frac{R-r+r}{r}\right) = \log\left(1+\frac{(R-r)}{r}\right) \sim \frac{(R-r)}{r}.(r \to R)\right].$$

Now we have

$$\frac{R-r}{r} = \frac{1}{q^{-1}\left(\frac{p(n)}{\overline{\sigma}}\right)^{1/\rho}}.$$

Then for r sufficiently close to R and large n, we get

$$\frac{\log\left(|a_n|n^{2\alpha+1}R^n\right)}{n} \leq \frac{1}{q^{-1}\left(\frac{p(n)}{\overline{\sigma}}\right)^{1/\rho}}$$

or

$$q^{-1}\left(\frac{p(n)}{\overline{\sigma}}\right)^{1/\rho} \leq \frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}$$

or

$$\frac{p(n)}{\overline{\sigma}} \leq \left\{q\left(\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho}$$

or

$$\frac{p(n)}{\left\{q\left(\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho}} \leq \overline{\sigma} = \sigma+\varepsilon.$$

Proceeding to limit supremum as $n \to \infty$, we get

$$\sigma(p,q) \geq T(p,q).$$

The result is obviously holds for $\sigma(p,q) = \infty$. Hence the proof is complete.

Remark 2.2. For $p(x) = q(x) = x$, Theorem 2.2 gives part (iii) of [20, Thm. 2].

3 Generalized $(p, q)-$Growth and Approximation of Pseudoanalytic Functions

In this section we obtain the generalized order and generalized type in terms of approximation errors defined by (1.5).

First we prove

Lemma 3.1. Let the pseudoanalytic function PAF $F \in P(D_R)$, has the series expansion

$$F(re^{i\theta}) = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta}), re^{i\theta} \in D_R,$$

then

$$\limsup_{n\to\infty} \frac{p(n)}{\left\{q\left(\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}} = \limsup_{n\to\infty} \frac{p(n)}{\left\{q\left(\frac{n}{\log(E_n(F)n^{2\alpha+1}R^n)}\right)\right\}}.$$

and

$$\limsup_{n\to\infty} \frac{p(n)}{\left\{q\left(\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho(p,q)}} = \limsup_{n\to\infty} \frac{p(n)}{\left\{q\left(\frac{n}{\log(E_n(F)n^{2\alpha+1}R^n)}\right)\right\}^{\rho(p,q)}}.$$

Proof. Using the identity [20, eq. 12]

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} [E_n(F)]^{1/n},$$

with the generalized $(p, q)-$order and generalized $(p, q)-$type calculation in Theorem 2.1 and Theorem 2.2, the proof is immediate.

Finally, we have the following theorem.

Theorem 3.1. Let F be pseudoanalytic function on $P(D_R)(R > 1)$. Then

(i) The generalized $(p, q)-$order $\rho(p, q)$ of F is

$$\rho(p,q) = \limsup_{n\to\infty} \frac{p(n)}{q\left(\frac{n}{\log(E_n(F)n^{2\alpha+1}R^n)}\right)}.$$

(ii) The generalized $(p, q)-$type of F is $T(p, q)$, if and only if,

$$T(p,q) = \limsup_{n\to\infty} \frac{p(n)}{\left\{q\left(\frac{n}{\log(E_n(F)n^{2\alpha+1}R^n)}\right)\right\}^{\rho(p,q)}}$$

when $0 < \rho(p, q) < \infty$.

Remark 3.1. For $p(x) = q(x) = \log x$ is the classical case for order i.e., part (i) of [20, Thm. 3] and $p(x) = q(x) = x$ is the classical case for type i.e., part (ii) of [20, Thm. 3].

Conclusions.The GBASP are natural extensions of harmonic or analytic functions. The Euler-Poisson-Darboux equation, arising in gas dynamics, is viewed in terms of real valued regular solution to the generalized biaxially symmetric potential equation after a suitable transformation. The GBASP then suggest generalization of analytic functions and have a variety of physical interpretations.The study of pseudoanalytic functions include the GBASP, therefore our study has a variety of applications.

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Competing interests

The authors declare that no competing interests exist.

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