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Extended Vector Equilibrium Problem

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Abstract

In this paper, we consider an extended vector equilibrium problem and prove some existence results in the setting of Hausdorff topological vector spaces and reflexive Banach spaces. Our results extend and improve some known results in the literature. Some examples are given.

Keywords: Vector equilibrium problem; Existence; Algorithm; Convexity

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1 Introduction

Let X be a Hausdorff topological vector space, K be a nonempty closed convex subset of X and let $f : K \times K \to R$ be a mapping with $f(x, x) = 0$, for all $x \in K$. Then the equilibrium problem is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \ \forall \, y \in K. \tag{1.1}$$

Problems like (1.1) were initially studied by Ky Fan [1] and Brezis et al. [2]. It was Blum and Oettli [3], who used the term equilibrium problem for the first time. Equilibrium problems include variational inequality problems as well as fixed point problems, optimization problems, complementarity problems, saddle point problems and Nash equilibrium problems as special cases, for more details, see, [4–9]. Equilibrium problems provide us a systematic framework to study a wide class of problems arising in finance, economics and operations research etc.. General equilibrium problems have been extended to the case of vector-valued bi-functions, known as vector equilibrium problems, see for example, [10–15].

Let X and Y be two Hausdorff topological vector spaces, K be a nonempty closed convex subset of X and C be a pointed closed convex cone in Y with $intC \neq \emptyset$. Given a vector valued mapping $f : K \times K \to Y$, the vector equilibrium problem consists of finding $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -intC, \ \forall \, y \in K. \tag{1.2}$$

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The vector equilibrium problems have been studied by many researchers and proven to be significant in the study of vector optimization, vector variational inequalities and vector complementarity problems, for more details, see, [16–21].

For comprehensive bibliography, we refer to Daniele et al. [22] and Giannessi [23] and references therein.

Motivated by the applications of vector equilibrium problems, in this paper, we introduce and study an extended vector equilibrium problem and prove some existence results in the setting of Hausdorff topological vector spaces and reflexive Banach spaces. Some special cases are also discussed.

2 Preliminaries

We need the following definitions and results for the presentation of this paper.

Definition 2.1. The Hausdorff topological vector space Y is said to be an ordered space denoted by (Y, C) if ordering relations are defined in Y by a pointed closed convex cone C of Y as follows:

$$\text{for all } x, y \in Y, y \leqq x \Leftrightarrow x - y \in C,$$

$$\text{for all } x, y \in Y, y \leq x \Leftrightarrow x - y \in C \setminus \{0\},$$

$$\text{for all } x, y \in Y, y \nleq x \Leftrightarrow x - y \notin C \setminus \{0\}.$$

If the interior of C, $intC \neq \emptyset$, then the weak ordering relations in Y are also defined as follows:

$$\text{for all } x, y \in Y, y < x \Leftrightarrow x - y \in intC,$$

$$\text{for all } x, y \in Y, y \not< x \Leftrightarrow x - y \notin intC.$$

Throughout this paper, unless otherwise specified, we assume that (Y, C) is an ordered Hausdorff topological vector space with $intC \neq \emptyset$.

Definition 2.2. Let K be a nonempty convex subset of a topological vector space X. A set-valued mapping $A : K \to 2^X$ is said to be KKM mapping, if for each finite subset $\{x_1, x_2, ..., x_n\}$ of K, $Co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} A(x_i)$, where $Co\{x_1, x_2, \cdots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \cdots, x_n\}$.

The following KKM-Fan theorem is important for us to prove the existence results of this paper.

Theorem 2.1. *(KKM-Fan Theorem) Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $A : K \to 2^X$ be a KKM mapping such that $A(x)$ is closed for all $x \in K$ and $A(x)$ is compact for at least one $x \in K$, then*

$$\bigcap_{x \in K} A(x) \neq \emptyset.$$

Let X and Y be the Hausdorff topological vector spaces and K be a nonempty closed convex subset of X. Let C be a pointed closed convex cone in Y with $intC \neq \emptyset$. Let $g : K \times X \to Y$ be a vector valued mapping and $\eta : K \times K \to X$ be a mapping. We introduce the following extended vector equilibrium problem.

Find $x_0 \in K$ such that for all $z \in K$ and $\lambda \in (0, 1]$

$$g(\lambda x_0 + (1 - \lambda)z, \eta(y, x_0)) \notin -intC, \ \forall\, y \in K. \tag{2.1}$$

If $\lambda = 1$ and $\eta(y, x_0) = y \in X$, the problem (2.1) reduces to the vector equilibrium problem of finding $x_0 \in K$ such that

$$g(x_0, y) \notin -intC, \ \forall\, y \in K. \tag{2.2}$$

Problem (2.2) was introduced and studied by Tan and Tinh [24].

In addition, if $Y = R$ and $C = R_+$, then problem (2.1) reduces to the equilibrium problem of finding $x_0 \in K$ such that

$$g(x_0, y) \geq 0, \ \forall\, y \in K. \tag{2.3}$$

Problem (2.3) was introduced and studied by Blum and Oettli [3].

The fact that problem (2.1) is much more general than many existing equilibrium, vector equilibrium problems, etc., motivate us to study Extended vector equilibrium problem (2.1).

Definition 2.3. Let X, Y be the Hausdorff topological vector spaces, K be a nonempty closed convex subset of X and C be a closed convex pointed cone in Y with $intC \neq \emptyset$. Let $\eta : K \times K \to X$ and $g : K \times X \to Y$ be the mappings. Then g is said to be

(i) η-monotone with respect to C, if and only if for all $x, y, z \in K$,

$$g(\lambda x + (1-\lambda)z, \eta(y, x)) + g(\lambda y + (1-\lambda)z, \eta(x, y)) \in -C;$$

(ii) η-hemicontinuous, if and only if for all $x, y \in K$, $t \in [0, 1]$, the mapping $t \to g(ty + (1 - t)x, \eta(y, x))$ is continuous at 0^+;

(iii) η-pseudomonotone, if and only if for all $x, y, z \in K$,

$$g(\lambda x + (1-\lambda)z, \eta(y, x)) \notin -intC \ \text{ implies } \ g(\lambda y + (1-\lambda)z, \eta(y, x)) \notin -intC;$$

(iv) η-generally convex, if and only if for all $x, y, z, w \in K$,

$$g(z, \eta(x, w)) \notin -intC \text{ and } g(z, \eta(y, w)) \notin -intC \ \text{ imply } \ g(z, \eta(\lambda x + (1-\lambda)y, w)) \notin -intC.$$

In support of Definition 2.3, we have the following examples.

Example 2.1. *Let* $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2$*, and* $C = \{(x, y) : x \leq 0, y \leq 0\}$.
Let $g : K \times X \to Y$ *and* $\eta : K \times K \to X$ *be the mappings such that*

$$g(x, y) = (y, x^2), \ \forall\, x, y \in K,$$
$$\text{and } \ \eta(x, y) = x^2 + y^2, \ \forall\, x, y \in K.$$

$$\begin{aligned}\text{Then,} \quad & g(\lambda x + (1-\lambda)z, \eta(y, x)) + g(\lambda y + (1-\lambda)z, \eta(x, y)) \\ &= (\eta(y, x), (\lambda x + (1-\lambda)z)^2) + (\eta(x, y), (\lambda y + (1-\lambda)z)^2) \\ &= (y^2 + x^2, (\lambda x + (1-\lambda)z)^2) + (x^2 + y^2, (\lambda y + (1-\lambda)z)^2) \in -C\end{aligned}$$

i.e., $g(\lambda x + (1-\lambda)z, \eta(y, x)) + g(\lambda y + (1-\lambda)z, \eta(x, y)) \in -C$.
Hence, g *is* η*-monotone with respect to* C.

Example 2.2. *Let* $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2$*, and* $C = \{(x, y) : x \leq 0, y \leq 0\}$*. Let* $F : [0, 1] \to Y$ *be a mapping such that*

$$F(t) = g(ty + (1-t)x, \eta(y, x)), \ \forall\, t \in [0, 1].$$

Let $g : K \times X \to Y$ *and* $\eta : K \times K \to X$ *be the mappings such that*

$$g(x, y) = (y, x^2), \ \forall\, x, y \in K,$$

$$\text{and } \ \eta(x, y) = x^2 + y^2, \ \forall\, x, y \in K.$$

$$\text{Then,} \quad F(t) = g(ty + (1-t)x, \eta(y,x)) = (\eta(y,x), (ty + (1-t)x)^2)$$
$$= (y^2 + x^2, (ty + (1-t)x)^2),$$

which implies that $t \to g(ty + (1-t)x, \eta(y,x))$ *is continuous at* 0^+.
Hence, g *is* η*-hemicontinuous.*

Example 2.3. *Let* $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2$*, and* $C = \{(x,y) : x \geq 0, y \geq 0\}$.
Let $g : K \times X \to Y$ *and* $\eta : K \times K \to X$ *be the mappings such that*

$$g(x,y) = (y, -x^2), \; \forall \, x, y \in K,$$

$$\text{and} \;\; \eta(x,y) = x - 2y, \; \forall \, x, y \in K.$$

$$\text{Then,} \quad g(\lambda x + (1-\lambda)z, \eta(y,x)) = (\eta(y,x), -(\lambda x + (1-\lambda)z)^2)$$
$$= ((y - 2x), -(\lambda x + (1-\lambda)z)^2) \notin -intC$$

implies $y \geq 2x$*, so it follows that*

$$g(\lambda y + (1-\lambda)z, \eta(y,x)) = (\eta(y,x), -(\lambda y + (1-\lambda)z)^2)$$
$$= ((y - 2x), -(\lambda y + (1-\lambda)z)^2) \notin -intC$$

Hence, g *is* η*-pseudomonotone with respect to* C.

Example 2.4. *Let* $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2$*, and* $C = \{(x,y) : x \geq 0, y \leq 0\}$.
Let $g : K \times X \to Y$ *and* $\eta : K \times K \to X$ *be the mappings such that*

$$g(x,y) = (y, x^2), \; \forall \, x, y \in K,$$

$$\text{and} \;\; \eta(x,y) = x - y, \; \forall \, x, y \in K.$$

$$\text{Then,} \quad g(z, \eta(x,w)) = (\eta(x,w), z^2)$$
$$= (x - w, z^2) \notin -intC,$$

implies $x \geq w$*, and*

$$g(z, \eta(y,w)) = (\eta(y,w), z^2)$$
$$= (y - w, z^2) \notin -intC,$$

implies $y \geq w$*, so it follows that*

$$g(z, \eta(\lambda x + (1-\lambda)y, w)) = (\eta(\lambda x + (1-\lambda)y, w), z^2)$$
$$= (\lambda x + (1-\lambda)y - w, z^2)$$
$$= (\lambda x + (1-\lambda)y - w + \lambda w - \lambda w, z^2)$$
$$= (\lambda(x - w) + (1-\lambda)(y - w), z^2) \notin -intC$$

implies $g(z, \eta(\lambda x + (1-\lambda)y, w)) \notin -intC$.
Hence, g *is* η*-generally convex.*

Definition 2.4. A mapping $\eta : K \times K \to X$ is said to be affine in the first argument, if and only if for all $x, y, z \in K$ and $t \in [0,1]$,

$$\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z).$$

Similarly one can define the affine property of η with respect to the second argument.

3 Existence Results

We prove the following equivalence lemma which we need for the proof of our main results.

Lemma 3.1. *Let X be a Hausdorff topological vector space, K be a closed convex subset of X and (Y, C) be an ordered Hausdorff topological vector space with $intC \neq \emptyset$. Let $g : K \times X \to Y$ be a vector valued mapping which is η-monotone with respect to C, positive homogeneous in the second argument and η-hemicontinuous. Let $\eta : K \times K \to X$ be a continuous and affine mapping in the first argument such that $\eta(x, x) = 0$ and $\eta(x, y) = -\eta(y, x)$, for all $x, y \in K$. Then for all $z \in K$ and $\lambda \in (0, 1]$, the following statements are equivalent.*

Find $x_0 \in K$ such that

(i) $g(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)) \notin -intC,\ \forall\, y \in K;$

(ii) $g(\lambda y + (1-\lambda)z, \eta(y, x_0)) \notin -intC,\ \forall\, y \in K.$

Proof. $(i) \Rightarrow (ii)$. For all $z \in K$ and $\lambda \in (0, 1]$, let x_0 be a solution of (i), then we have

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)) \notin -intC.$$

Since g is η-monotone with respect to C, we have

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)) + g(\lambda y + (1-\lambda)z, \eta(x_0, y)) \in -C$$

and as $\eta(x, y) = -\eta(y, x)$, we have

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)) \in -C + g(\lambda y + (1-\lambda)z, \eta(y, x_0)). \tag{3.1}$$

Suppose to the contrary that (ii) is false. Then there exists $y \in K$ such that

$$g(\lambda y + (1-\lambda)z, \eta(y, x_0)) \in -intC.$$

By (3.1), we obtain

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)) \in -C - \text{int } C \subset -intC,$$

which contradicts (i). Thus $(i) \Rightarrow (ii)$.
$(ii) \Rightarrow (i)$. Conversely, suppose that (ii) holds. Then $x_0 \in K$ satisfies

$$g(\lambda y + (1-\lambda)z, \eta(y, x_0)) \notin -intC,\ \forall\, y \in K.$$

For each $y \in K, t \in [0, 1]$, we let $y_t = ty + (1-t)x_0$. Since K is convex, $y_t \in K$. Then we have

$$g(\lambda y_t + (1-\lambda)z, \eta(y_t, x_0)) \notin -intC.$$

Since η is affine in the first argument and $\eta(x_0, x_0) = 0$, we have

$$g(\lambda(ty + (1-t)x_0)) + (1-\lambda)z, t\eta(y, x_0)) \notin -intC.$$

By positive homogeneity of g in the second argument, we obtain

$$g(\lambda(ty + (1-t)x_0)) + (1-\lambda)z, \eta(y, x_0)) \notin -intC.$$

As g is η-hemicontinuous, let $t \to 0^+$, we have

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)) \notin -intC,\ \forall\, y \in K.$$

Thus $(ii) \Rightarrow (i)$. This completes the proof. □

Theorem 3.1. *Let X be a Hausdorff topological vector space and let K be a nonempty compact and convex subset of X, and (Y, C) be an ordered Hausdorff topological vector space with $intC \neq \emptyset$. Let $g : K \times X \to Y$ be a vector valued mapping which is η-monotone with respect to C, positive homogeneous in the second argument and η-hemicontinuous and let the mapping $x \to g(\lambda y + (1-\lambda)x, \eta(y, x))$ be continuous. Let $\eta : K \times K \to X$ be a continuous and affine in both the arguments such that $\eta(x, x) = 0$ and $\eta(x, y) = -\eta(y, x)$, for all $x, y \in K$. Then, problem (2.1) admits a solution, that is, for all $z \in K$ and $\lambda \in (0, 1]$, there exists $x_0 \in K$ such that*

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)) \notin -intC, \ \forall\, y \in K.$$

Proof. For each $y \in K$, we define

$$M(y) = \{x \in K : g(\lambda x + (1-\lambda)z, \eta(y, x)) \notin -intC\};$$

$$S(y) = \{x \in K : g(\lambda y + (1-\lambda)z, \eta(y, x)) \notin -intC\}.$$

Clearly $M(y) \neq \emptyset$, as $y \in M(y)$. We divide the proof into three steps.
Step 1 We claim that $M : K \to 2^K$ is a KKM-mapping. If M is not a KKM-mapping, then there exists $x \in \text{Co}\{y_1, y_2, \cdots, y_n\}$ such that for all $t_i \in [0, 1], i = 1, 2, \cdots, n$ with $\sum\limits_{i=1}^{n} t_i = 1$, we have

$$x = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} M(y_i).$$

Thus, we have

$$g(\lambda x + (1-\lambda)z, \eta(y_i, x)) \in -intC, i = 1, 2, \cdots, n.$$

Since η is affine in the second argument and $\eta(y_i, y_i) = 0$, we have

$$g(\lambda x + (1-\lambda)z, \eta(y_i, \sum_{i=1}^{n} t_i y_i)) = \sum_{i=1}^{n} t_i g(\lambda x + (1-\lambda)z, \eta(y_i, y_i)) \in -intC.$$

It follows that $0 \in -intC$, which is a contradiction. Thus M is a KKM-mapping.
Step 2 $\bigcap\limits_{y\in K} M(y) = \bigcap\limits_{y\in K} S(y)$ and S is also a KKM-mapping.
If $x \in M(y)$, then $g(\lambda x + (1-\lambda)z, \eta(y, x)) \notin -intC$. By the η-monotonicity of g with respect to C and using the fact that $\eta(x, y) = -\eta(y, x)$, we have

$$g(\lambda x + (1-\lambda)z, \eta(y, x)) \in g(\lambda y + (1-\lambda)z, \eta(y, x)) - C. \tag{3.2}$$

Suppose that $x \notin S(y)$. Then, we have

$$g(\lambda y + (1-\lambda)z, \eta(y, x)) \in -intC.$$

It follows from (3.2) that

$$g(\lambda x + (1-\lambda)z, \eta(y, x)) \in -intC - C \subset -intC,$$

which contradicts that $x \in M(y)$. Therefore $x \in S(y)$, that is, $M(y) \subset S(y)$. Then

$$\bigcap_{y\in K} M(y) \subset \bigcap_{y\in K} S(y).$$

On the other hand, Suppose that $x \in \bigcap\limits_{y\in K} S(y)$. We have

$$g(\lambda y + (1-\lambda)z, \eta(y, x)) \notin -intC, \ \forall\, y \in K.$$

By Lemma 3.1, we have

$$g(\lambda x + (1-\lambda)z, \eta(y,x)) \notin -intC, \ \forall\ y \in K.$$

That is, $x \in \bigcap_{y\in K} M(y)$. Hence,
$\bigcap_{y\in K} M(y) \supset \bigcap_{y\in K} S(y)$. So, $\bigcap_{y\in K} M(y) = \bigcap_{y\in K} S(y)$.

Also $\bigcap_{y\in K} S(y) \neq \emptyset$, since $y \in S(y)$. From above, we know that $M(y) \subset S(y)$ and by Step 1, we know that M is a KKM-mapping. Thus S is also a KKM-mapping.

Step 3 For all $y \in K, S(y)$ is closed.
Let $\{x_n\}$ be a sequence in $S(y)$ such that $\{x_n\}$ converges to $x \in K$. Then

$$g(\lambda y + (1-\lambda)z, \eta(y,x_n)) \notin -intC, \text{ for all } n.$$

Since the mapping $x \to g(\lambda y + (1-\lambda)z, \eta(y,x))$ is continuous, we have

$$g(\lambda y + (1-\lambda)z, \eta(y,x_n)) \to g(\lambda y + (1-\lambda)z, \eta(y,x)) \notin -intC.$$

We conclude that $x \in S(y)$, that is, $S(y)$ is a closed subset of a compact set K and hence compact. By KKM Theorem 2.1, $\bigcap_{y\in K} S(y) \neq \emptyset$ and also $\bigcap_{y\in K} M(y) \neq \emptyset$. Hence there exists $x_0 \in \bigcap_{y\in K} M(y) = \bigcap_{y\in K} S(y)$, that is, there exists $x_0 \in K$ such that

$$g(\lambda x_0 + (1-\lambda)z, \eta(y,x_0)) \notin -intC, \ \forall\ y,z \in K \text{ and } \lambda \in (0,1],$$

thus, x_0 is a solution of problem (2.1). □

In support of Theorem 3.1, we give the following example.

Example 3.1. *Let $X = Y = \mathbb{R}, K = \mathbb{R}_+$, and $C = \{x : x \geq 0\}$.*
Let $g : K \times X \to Y$ and $\eta : K \times K \to X$ be the mappings such that

$$g(x,y) = xy, \ \forall\ x,y \in K,$$

$$\textit{and}\ \ \eta(x,y) = 2x - 2y, \ \forall\ x,y \in K.$$

Then,

(i) For any $x,y,z \in K$,

$$\begin{aligned} g(\lambda x + (1-\lambda)z, \eta(y,x)) &+ g(\lambda y + (1-\lambda)z, \eta(x,y)) \\ &= [\lambda x + (1-\lambda)z]\eta(y,x) + [\lambda y + (1-\lambda)z]\eta(x,y) \\ &= [\lambda x + (1-\lambda)z](2y-2x) + [\lambda y + (1-\lambda)z](2x-2y) \\ &= -2\lambda(y-x)^2 \in -C, \end{aligned}$$

i.e., g is η-monotone with respect to C.

(ii) For any $r > 0$,
$g(x,ry) = xry = r(xy) = rg(x,y)$
i.e., g is positive homogeneous in the second argument.

(iii) Let $F : [0,1] \to Y$ be a mapping such that

$$F(t) = g(ty + (1-t)x, \eta(y,x)), \ \forall\ t \in [0,1].$$

$$\begin{aligned} \text{Then,}\quad F(t) = g(ty + (1-t)x, \eta(y,x)) &= [ty + (1-t)x]\eta(y,x) \\ &= [ty + (1-t)x](2y-2x), \end{aligned}$$

which is a continuous mapping.
i.e., $t \to g(ty + (1-t)x, \eta(y,x))$ is continuous at 0^+.
Hence, g is η-hemicontinuous.

(iv) Let $G : K \to Y$ be a mapping such that

$$G(x) = g(\lambda y + (1-\lambda)x, \eta(y,x)),\ \forall\, x \in K.$$

$$\text{Then,} \quad G(x) = g(\lambda y + (1-\lambda)x, \eta(y,x)) = [\lambda y + (1-\lambda)x]\eta(y,x) = [\lambda y + (1-\lambda)x](2y-2x),$$

which implies that $x \to g(\lambda y + (1-\lambda)x, \eta(y,x))$ is continuous mapping.

(v) $\eta(x,y) = 2x - 2y$. Then,

$$\begin{aligned}\eta(\lambda x_1 + (1-\lambda)x_2, y) &= 2\lambda x_1 + 2(1-\lambda)x_2 - 2y \\ &= 2\lambda x_1 + 2(1-\lambda)x_2 - 2y + 2\lambda y - 2\lambda y \\ &= \lambda(2x_1 - 2y) + (1-\lambda)(2x_2 - 2y) \\ &= \lambda\eta(x_1,y) + (1-\lambda)\eta(x_2,y)\end{aligned}$$

i.e., η is affine in the first argument.

Similarly one can show that η is affine in the second argument.

(vi) By the definition of η, we have
$\eta(x,y) = (2x-2y) = -(2y-2x) = -\eta(y,x)$ *and* $\eta(x,x) = 0$.

Hence, all the conditions of Theorem 3.1 are satisfied.
In addition,

$$\begin{aligned}g(\lambda x_0 + (1-\lambda)z, \eta(y,x_0)) &= [\lambda x_0 + (1-\lambda)z]\eta(y,x_0) \\ &= [\lambda x_0 + (1-\lambda)z](2y - 2x_0) \\ &= 2[\lambda x_0 + (1-\lambda)z](y - x_0) \notin -intC, \text{ for } x_0 \le y,\end{aligned}$$

Thus, it follows that x_0 is a solution of problem (2.1) for all $z \in K$ and $\lambda \in (0,1]$.

Corollary 3.1. *Let K be a nonempty compact convex subset of X and (Y,C) be an ordered topological vector space with $intC \neq \emptyset$. Let $g : K \times X \to Y$ be a vector valued mapping which is η-pseudomonotone with respect to C and let $\eta : K \times K \to X$ be a continuous and affine mapping in both the arguments such that $\eta(x,x) = 0$, for all $x \in K$. Let the mapping $x \to g(\lambda x + (1-\lambda)z, \eta(y,x))$ be continuous. Then problem (2.1) is solvable.*

Proof. By step 1 of Theorem 3.1, it follows that M is a KKM-mapping. Also it follows from η-pseudomonotonicity of g that $M(y) \subset S(y)$, thus S is also a KKM-mapping. By step (3) of Theorem 3.1, the conclusion follows. □

Theorem 3.2. *Let X be a reflexive Banach space, (Y,C) be an ordered topological vector space with $intC \neq \emptyset$. Let K be a nonempty, bounded and convex subset of X. Let $g : K \times X \to Y$ be a vector valued mapping which is η-monotone with respect to C, positive homogeneous in the second argument, η-hemicontinuous and η-generally convex on K. Let $\eta : K \times K \to X$ be a continuous and affine mapping in both the arguments such that $\eta(x,x) = 0$ and $\eta(x,y) = -\eta(y,x)$, for all $x, y \in K$. Then, problem (2.1) is solvable, that is, for all $z \in K$ and $\lambda \in (0,1]$, there exists $x_0 \in K$ such that*

$$g(\lambda x_0 + (1-\lambda)z, \eta(y,x_0)) \notin -intC,\ \forall\, y \in K.$$

Proof. For each $y \in K$, let

$$M(y) = \{x \in K : g(\lambda x + (1-\lambda)z, \eta(y,x)) \notin -intC\},$$

$$S(y) = \{x \in K : g(\lambda y + (1-\lambda)z, \eta(y,x)) \notin -intC\}, \text{ for all } z \in K \text{ and } \lambda \in (0,1].$$

From the proof of the Theorem 3.1, we know that $S(y)$ is closed and S is a KKM-mapping. We also know that

$$\bigcap_{y \in K} M(y) = \bigcap_{y \in K} S(y).$$

Since K is a bounded, closed and convex subset of a reflexive Banach space X, therefore K is weakly compact.

Now, we show that $S(y)$ is convex. Suppose that $y_1, y_2 \in S(y)$ and $t_1, t_2 \geq 0$ with $t_1 + t_2 = 1$. Then

$$g(\lambda y + (1-\lambda)z, \eta(y, y_i)) \notin -intC, \quad i = 1, 2.$$

Since g is η-generally convex, we have

$$g(\lambda y + (1-\lambda)z, \eta(y, t_1 y_1 + t_2 y_2)) \notin -intC,$$

that is, $t_1 y_1 + t_2 y_2 \in S(y)$, which implies that $S(y)$ is convex. Since $S(y)$ is closed and convex, $S(y)$ is weakly closed.

As S is a KKM-mapping, $S(y)$ is weakly closed subset of K, therefore $S(y)$ is weakly compact. By KKM Theorem 2.1, there exists $x_0 \in K$ such that $x_0 \in \bigcap_{y \in K} M(y) = \bigcap_{y \in K} S(y) \neq \emptyset$. That is, there exists $x_0 \in K$ such that

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)) \notin -intC, \ \forall\, y, z \in K \text{ and } \lambda \in (0, 1].$$

Hence problem (2.1) is solvable. □

4 Conclusions

In this paper, we have extended the classical vector equilibrium problem. Some existence results are proved in the setting of Hausdorff topological vector spaces and reflexive Banach spaces. The results of this paper can be viewed as generalizations of many known equilibrium problems and can be used for further research in this area.

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Competing Interests

The authors declare that no competing interests exist.

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