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# On Solvability of the Neumann Boundary Value Problem for Non-homogeneous Biharmonic Equation

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## Abstract

In this work we investigate the Neumann boundary value problem in the unit ball for a non-homogeneous biharmonic equation. It is well known, that even for the Poisson equation this problem does not have a solution for an arbitrary smooth right hand side and boundary functions; it follows from the Green formula, that these given functions should satisfy a condition called the solvability condition. In the present paper these solvability conditions are found in an explicit form for the natural generalization of the Neumann problem for the non-homogeneous biharmonic equation. The method used is new for these type of problems. We first reduce this problem to the Dirichlet problem, then use the Green function of the Dirichlet problem recently found by T. Sh. Kal'menov and D. Suragan

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## 1 Introduction

Let $\Omega = \{x \in R^n : |x| < 1\}$ be the unit ball, $\partial\Omega = \{x \in R^n : |x| = 1\}$ be the unit sphere.

Consider on the domain $\Omega$ the Neumann boundary value problem:

$$\Delta u(x) = g(x), \;\; x \in \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu}(x) = \psi(x), \;\; x \in \partial\Omega, \tag{1.2}$$

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where $\nu$ is the unit outer normal vector to sphere $\partial\Omega$, $g(x)$ and $\psi(x)$ are given functions; we always suppose that the right hand side and the boundary functions are sufficiently smooth and from here on do not pay any attention to their smoothness.

It follows from the Green formula, that unlike to the Dirichlet problem, this problem does not have solutions for arbitrary (even, as we supposed, smooth) functions $g(x)$ and $\psi(x)$ (see, for example, [1]); these functions should satisfy the necessary and sufficient solvability condition:

$$\int_{\Omega} g(x)dx = \int_{\partial\Omega} \psi(x)dS_x.$$

In the paper [2] by B.E. Kanguzhin and B.D. Koshanov the general Nuemann problem for the polyharmonic equation:

$$(-\Delta)^m u(x) = g(x),\ x \in \Omega, \tag{1.3}$$

$$\frac{\partial^{k_j} u}{\partial \nu^{k_j}}(x) = \psi_{k_j}(x),\ j = 1, ..., m,\ x \in \partial\Omega, \tag{1.4}$$

was considered, where $m$ is any positive integer, $0 < k_i < k_j \le 2m-1,\ 1 \le i < j \le m.$ The authors found a solvability condition for the problem (1.3), (1.4) (see [2], Theorem 4.2). This condition follows from equality to zero of the determinant of a $m \times m$ matrix, one column of which consists of integrals $\int_{\partial\Omega}[\psi_{k_j}(x) - \partial^{k_j}/\partial\nu^{k_j}(\varepsilon_{m,n} * g(x))]dS_x$, $\varepsilon_{m,n} = d_{m,n}|x|^{2m-n}$ and $d_{m,n}$ is a constant. Note, the equation which one has as a result is very difficult to verify.

There are no work yet where these conditions are simplified in the general case. But the authors of the paper [3] considered a particular case of the problem (1.3), (1.4), i.e. $m = 2$, $k_1 = 1$ and $k_2 = 2$ and presented the solvability condition in a different form, which could be easily verified:

$$\int_{\Omega} \frac{1-|x|^2}{2} g(x)dx = \int_{\partial\Omega} [\psi_2(x) - \psi_1(x)]dS_x.$$

This result was generalized for an arbitrary $m$ and $k_j = j$ by authors of this paper [4] (see also [5]).

If we compare the Neumann problem (1.1), (1.2) with the Diriclet problem (i.e. $u(x) = \psi(x), x \in \partial\Omega$) for the same equation, then we can note that in the boundary we have different functions (i.e. the function itself and its derivative).

If we generalize in the same way the Neumann problem for the biharmonic equation (i.e. $m = 2$), then we should take $k_1 = 2$ and $k_2 = 3$ (instead of $k_1 = 1$ and $k_2 = 2$ as in [3]). So the more natural generalization of the Neymann problem for the biharmonic equation has the form:

$$\Delta^2 u(x) = g(x),\ x \in \Omega, \tag{1.5}$$

$$\frac{\partial^{k+1} u}{\partial \nu^{k+1}}(x) = \psi_k(x),\ k = 1, 2,\ x \in \partial\Omega, \tag{1.6}$$

Note in the Dirichlet problem for the equation (1.3) with $m = 2$ one has $k_1 = 0$ and $k_2 = 1$ in (1.4).

A function $u(x) \in C^4(\Omega) \cap C^3(\overline{\Omega})$ is called to be a solution of problem (1.5), (1.6), if it satisfies (1.5), (1.6) in classical sense.

It should be noted that many the Neumann type problems (different from (1.6) for equation (1.5) were considered in the paper by [6] (see also references therein). The author found the necessary and sufficient solvability conditions in a very simple form.

The main goal of the present paper is to find solvability conditions for the Neumann problem (1.5), (1.6). In Theorem 4.1 (see also Theorem 5.1) we show that in fact there are $n+1$ such conditions. Since the homogeneous problem has $n+1$ solutions, then this fact states that the Neumann problem (1.5), (1.6) is correct in the Fredholm sense.

It should be noted that in our study of problem (1.5), (1.6) the Green function of the Dirichlet problem for equation (1.5) is essentially used. In the paper [7] a similar method was used in the solution of the boundary value problem for the Poisson equation with the boundary operator of fractional-order.

## 2 Properties of some integro-differential operators

Let $u(x)$ be a sufficiently smooth function in $\Omega$. Consider the following operators

$$\Gamma_c[u](x) = \left(r\frac{\partial}{\partial r} + c\right)u(x),\ \Gamma_c^{-1}[u](x) = \int_0^1 t^{c-1}u(tx)dt, \tag{2.1}$$

where $r = |x|$ and $c > 0$ is a constant.

Note that in the class of harmonic functions in a ball the properties of operators $\Gamma_c$ and $\Gamma_c^{-1}$ have previously been studied in the paper [8].

**Lemma 2.1.** *Let $u(x)$ be a smooth function. Then for any $x \in \Omega$ one has*

$$\Gamma_c[\Gamma_c^{-1}[u]](x) = \Gamma_c^{-1}[\Gamma_c[u]](x) = u(x). \tag{2.2}$$

*Proof.* We have

$$\int_0^1 \frac{d}{dt}[t^c u(tx)]dt = \int_0^1 t^{c-1}\left[cu(tx) + t\frac{d}{dt}u(tx)\right]dt = \int_0^1 t^{c-1}\Gamma_c[u](tx)dt.$$

Therefore,

$$u(x) = \int_0^1 t^{c-1}\Gamma_c[u](tx)dt = \Gamma_c^{-1}[\Gamma_c[u]](x),$$

Hence the second equality of (2.2) is proved.

Now apply to the function $\Gamma_c^{-1}[u](x)$ operator $\Gamma_c$. Then

$$\Gamma_c[\Gamma_c^{-1}[u]](x) = \int_0^1 t^{c-1}\Gamma_c[u](tx)dt = u(x).$$

Hence the first equality in (2.2), and therefore the lemma is proved. □

**Corollary 2.2.** *Let $c_1$ and $c_2$ be positive numbers and $u(x)$ be a smooth function. Then for any $x \in \Omega$ one has*

$$\Gamma_{c_2}^{-1}\Gamma_{c_1}^{-1}[\Gamma_{c_1}\Gamma_{c_2}[u]](x) = \Gamma_{c_1}\Gamma_{c_2}[\Gamma_{c_2}^{-1}\Gamma_{c_1}^{-1}[u]](x) = u(x). \tag{2.3}$$

The following statement can be proved by direct calculation.

**Lemma 2.3.** *Let $u(x)$ be a smooth function. Then for any $x \in \Omega$ one has*

$$\Delta\Gamma_c[u](x) = \Gamma_{c+2}[\Delta u](x),$$

$$\Delta\Gamma_c^{-1}[u](x) = \Gamma_{c+2}^{-1}[\Delta u](x).$$

**Corollary 2.4.** *Let $u(x)$ be a smooth function. Then for any $x \in \Omega$ one has*

$$\Delta^2\Gamma_c[u](x) = \Gamma_{c+4}[\Delta^2 u](x), \tag{2.4}$$

$$\Delta^2\Gamma_c^{-1}[u](x) = \Gamma_{c+4}^{-1}[\Delta^2 u](x). \tag{2.5}$$

**Corollary 2.5.** *Let $c_1$ and $c_2$ be positive numbers and $u(x)$ be a smooth function. Then for any $x \in \Omega$ one has*

$$\Delta^2\Gamma_{c_1}[\Gamma_{c_2}[u]](x) = \Gamma_{c_1+4}[\Gamma_{c_2+4}[\Delta^2 u]](x),\ x \in \Omega, \tag{2.6}$$

$$\Delta^2\Gamma_{c_1}^{-1}[\Gamma_{c_2}^{-1}[u]](x) = \Gamma_{c_1}^{-1}[\Gamma_{c_2}^{-1}[\Delta^2 u]](x). \tag{2.7}$$

*Remark* 2.1. It is not hard to show that the equality (2.6) is true for any real numbers $c_1$ and $c_2$.

# 3 Some properties of the solutions of the Dirichlet problem

Let $v(x)$ be a solution of the Dirichlet problem

$$\begin{cases} \Delta^2 v(x) = g_1(x), & x \in \Omega, \\ \dfrac{\partial^{k-1} v}{\partial \nu^{k-1}}(x) = \varphi_k(x),\ k = 1, 2, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Let us first consider the case when $\varphi_1(x) \equiv \varphi_2(x) \equiv 0$. It is known (see, for example, [9, 10, 11]), that if $g_1(x)$ is sufficiently smooth function, then the solution of problem (3.1) exists, it is unique and has the form

$$v(x) = \int_\Omega G_{m,n}(x, y) g_1(y) dy, \tag{3.2}$$

where $G_{m,n}(x, y)$ is the Green function of the Dirichlet problem (3.1).

We note that the Green functions for many type of the Dirichlet and the Nuemann problems for equation (1.5) were constructed by [6] (see also the references therein).

We make use the following explicit form of the Green function [9]:

if $n$ is odd, or even and $n > 4$, then

$$G_{2,n}(x, y) = d_{2,n}\big[|x - y|^{4-n} - \big|x|y| - \frac{y}{|y|}\big|^{4-n} -$$

$$\big(2 - \frac{n}{2}\big)\big|x|y| - \frac{y}{|y|}\big|^{2-n}(1 - |x|^2)(1 - |y|^2)\big],$$

where

$$d_{2,n} = \frac{\Gamma(\frac{n}{2} - 2)}{\pi^{\frac{n}{2}} 4^2};$$

if $n = 2$ or $n = 4$, then

$$G_{2,n}(x, y) = d_{2,n}\big[|x - y|^{4-n}\left(\ln|x - y|^2 - \ln\left|x|y| - \frac{y}{|y|}\right|^2\right) +$$

$$\left|x|y| - \frac{y}{|y|}\right|^{2-n}(1 - |x|^2)(1 - |y|^2)\big]$$

where

$$d_{2,n} = \frac{(-1)^{2-n/2}}{2\pi^{\frac{n}{2}} 4^2 (2 - n/2)!}.$$

To use the explicit form of the Green function we shall deal only with the case $n$ is odd or even and $n > 4$, the other cases being exactly similar.

**Lemma 3.1.** *Let $g_1(x) = \Gamma_4[\Gamma_3[g]](x)$ and $\varphi_1(x) \equiv \varphi_2(x) \equiv 0$ in the Dirichlet problem (3.1). Let $v(x)$ be the unique solution of this problem. Then*

$$v(0) = \frac{1}{4\omega_n} \int_\Omega (1 - |y|^2) \Gamma_3[g](y) dy, \tag{3.3}$$

*where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ - the measure of the unit sphere.*

*Proof.* Let $v(x)$ be the solution of problem (3.1). Then it has the form (3.2). Therefore,

$$v(0) = d_{2,n} \int_\Omega \left[|y|^{4-n} - 1 + \left(2 - \frac{n}{2}\right)(1 - |y|^2)\right] g_1(y) dy.$$

If we denote $\rho = |y|$ and $\xi = \frac{y}{|y|}$, then the last integral can be rewritten as

$$v(0) = d_{2,n} \int_{|\xi|=1} \int_0^1 \rho^{n-1} \big[\rho^{4-n} - 1 + \frac{4-n}{2}(1-\rho^2)\big] g_1(\rho,\xi) d\rho d\xi =$$

$$\int_{|\xi|=1} I(\xi) d\xi.$$

Now we consider the inner integral $I(\xi)$. Noting that $g_1(\rho,\xi) = (\rho\frac{\partial}{\partial\rho} + 4)\Gamma_3[g](\rho,\xi)$, we introduce the following two integrals:

$$I_1(\xi) = \int_0^1 \big[4\rho^3 + \frac{4\cdot(2-n)}{2}\rho^{n-1} - \frac{4\cdot(4-n)}{2}\rho^{n+2}\big]\Gamma_3[g](\rho,\xi)d\rho,$$

$$I_2(\xi) = \int_0^1 \big[\rho^4 + \frac{2-n}{2}\rho^n - \frac{4-n}{2}\rho^{n+2}\big]\frac{\partial}{\partial\rho}\Gamma_3[g](\rho,\xi)d\rho.$$

Obviously, $I(\xi) = I_1(\xi) + I_2(\xi)$.

Integrating by parts in the integral $I_2(\xi)$ we obtain

$$I_2(\xi) = \int_0^1 \big[-4\rho^3 - \frac{(2-n)\cdot n}{2}\rho^{n-1} + \frac{(4-n)\cdot(n+2)}{2}\rho^{n+1}\big]\Gamma_3[g](\rho,\xi)d\rho.$$

Therefore

$$I(\xi) = I_1(\xi) + I_2(\xi) = \frac{(2-n)\cdot(4-n)}{2}\int_0^1 \rho^{n-1}[1-\rho^2]\Gamma_3[g](\rho,\xi)d\rho.$$

By virtue of the equality

$$d_{2,n} = \frac{1}{\omega_n}\cdot\frac{1}{2(n-2)(n-4)}$$

we have for $v(0)$ the following

$$v(0) = \frac{1}{4\omega_n}\int_{|\xi|=1}\int_0^1 \rho^{n-1}(1-\rho^2)\Gamma_3[g](\rho,\xi)d\rho d\xi$$

and going back to the Cartesian coordinate system, we finally obtain

$$v(0) = \frac{1}{4\omega_n}\int_\Omega (1-|y|^2)\Gamma_3[g](y)dy.$$

□

**Lemma 3.2.** *Let the conditions of Lemma 3.1 be satisfied. Then*

$$\frac{\partial v}{\partial x_j}(0) = \frac{n}{4\omega_n}\int_\Omega y_j(1-|y|^2)\Gamma_4[g](y)dy,\ j = 1,2,...,n. \tag{3.4}$$

*Proof.* It is not hard to verify, that

$$\frac{\partial}{\partial x_j}|x-y|^{4-n} = \frac{4-n}{2}|x-y|^{2-n}2(x_j-y_j)\Big|_{x=0} = -(4-n)|y|^{2-n}y_j,$$

$$\frac{\partial}{\partial x_j}\left|x|y| - \frac{y}{|y|}\right|^{4-n} = \frac{4-n}{2}\left|x|y|-\frac{y}{|y|}\right|^{2-n}2\left(x_j|y| - \frac{y_j}{|y|}\right)|y|\Big|_{x=0} = -(4-n)y_j,$$

and

$$\frac{\partial}{\partial x_j}\left[\left|x|y|-\frac{y}{|y|}\right|^{2-n}\left(1-|x|^2\right)\right] = \frac{2-n}{2}\left|x|y|-\frac{y}{|y|}\right|^{-n}2\left(x_j|y|-\frac{y_j}{|y|}\right)|y|\times$$

$$\left(1-|x|^2\right)+\left|x|y|-\frac{y}{|y|}\right|^{2-n}(-2x_j)\Bigg|_{x=0}=-(2-n)y_j.$$

Therefore

$$\frac{\partial v(0)}{\partial x_j}=\int\limits_{\Omega}\frac{\partial}{\partial x_j}G_{2,n}\left(0,y\right)g_1\left(y\right)dy=$$

$$-(4-n)d_{2,n}\int\limits_{\Omega}y_j\left[|y|^{2-n}-\frac{n}{2}-\frac{2-n}{2}|y|^2\right]g_1\left(y\right)dy.$$

Passing to the polar coordinate system $\rho=|y|$, $\xi=\frac{y}{|y|}$, we have

$$\frac{\partial v(0)}{\partial x_j}=-(4-n)d_{2,n}\int\limits_{|\xi|=1}\xi_j\int\limits_0^1\rho^n\left[\rho^{2-n}-\frac{n}{2}-\frac{2-n}{2}\rho^2\right]g_1\left(\rho,\xi\right)d\rho d\xi.$$

Now we consider the inner integral. Noting that $g_1(\rho,\xi)=\Gamma_4[\Gamma_3[g]](\rho,\xi)$, we introduce the following two integrals:

$$J_1=\int\limits_0^1\left[3\rho^2-\frac{3n}{2}\rho^n-\frac{3(2-n)}{2}\rho^{n+2}\right]\Gamma_4[g]\left(\rho,\xi\right)d\rho,$$

$$J_2=\int\limits_0^1\left[\rho^3-\frac{n}{2}\rho^{n+1}-\frac{(2-n)}{2}\rho^{n+3}\right]\frac{\partial}{\partial\rho}\Gamma_4[g]\left(\rho,\xi\right)d\rho.$$

Integrating by parts in $J_2$ gives

$$J_2=\int\limits_0^1\left[-3\rho^2+\frac{n(n+1)}{2}\rho^n+\frac{(2-n)(n+3)}{2}\rho^{n+2}\right]\Gamma_4[g]\left(\rho,\xi\right)d\rho.$$

Therefore

$$J_1+J_2=\int\limits_0^1\left[\frac{n(n-2)}{2}\rho^n+\frac{n(2-n)}{2}\rho^{n+2}\right]\Gamma_4[g]\left(\rho,\xi\right)d\rho=$$

$$\frac{n(n-2)}{2}\int\limits_0^1\left[\rho^n-\rho^{n+2}\right]\Gamma_4[g]\left(\rho,\xi\right)d\rho.$$

Hence

$$\frac{\partial v(0)}{\partial x_j}=\frac{n(n-2)(n-4)}{2}d_{2,n}\int\limits_{|\xi|=1}f_j(\xi)\int\limits_0^1\rho^n\left[1-\rho^2\right]\Gamma_4[g]\left(\rho,\xi\right)d\rho d\xi=$$

$$\frac{1}{\omega_n}\frac{n}{4}\int\limits_{|\xi|=1}\int\limits_0^1\rho^{n-1}\rho\xi_j\left[1-\rho^2\right]\Gamma_4[g]\left(\rho,\xi\right)d\rho d\xi.$$

Note $\rho\xi_j=y_j$. Therefore going back to the Cartesian coordinate system, we finally obtain

$$\frac{\partial v}{\partial x_j}(0)=\frac{n}{4\omega_n}\int\limits_{\Omega}y_j(1-|y|^2)\Gamma_4\left[g\right](y)dy.$$

□

Now we investigate the properties of the solution of problem (3.1) in case of $g_1(x) \equiv 0$.

**Lemma 3.3.** *Let $g_1(x) \equiv 0$ and $\varphi_1(x)$, $\varphi_2(x)$ be sufficiently smooth functions. Then the solution $v(x)$ of problem (3.1) satisfies the following conditions:*

$$v(0) = \frac{1}{2\omega_n} \int\limits_{\partial\Omega} [2\varphi_1(y) - \varphi_2(y)]\, dS_y, \tag{3.5}$$

$$\frac{\partial v(0)}{\partial x_j} = \frac{n}{2\omega_n} \int\limits_{\partial\Omega} y_j\, [3\varphi_1(y) - \varphi_2(y)]\, dS_y, j = 1, 2, ..., n. \tag{3.6}$$

*Proof.* Let $v(x)$ be the solution of problem (3.1) with the function $g_1(x) \equiv 0$. Making use of the Almansi formula (see, for example, [12], p.188) we write the solution of problem (3.1) as

$$v(x) = v_0(x) + (1 - |x|^2)v_1(x), \tag{3.7}$$

where $v_j(x)$ are harmonic functions in the ball $\Omega$.

Substituting the function (3.7) into the boundary condition of (3.1) we obtain two Dirichlet problems:

$$\begin{cases} \Delta v_0(x) = 0, & x \in \Omega, \\ v_0(x) = \varphi_1(x), & x \in \partial\Omega, \end{cases} \tag{3.8}$$

$$\begin{cases} \Delta v_1(x) = 0, & x \in \Omega, \\ v_1(x) = -\dfrac{1}{2}\big[\varphi_2(x) - \dfrac{\partial v_0}{\partial \nu}(x)\big] & x \in \partial\Omega. \end{cases} \tag{3.9}$$

We represent the solutions of these problems as the Poisson integrals:

$$v_0(x) = \frac{1}{\omega_n} \int\limits_{\partial\Omega} \frac{1 - |x|^2}{|x - y|^n} \varphi_1(y) dS_y,$$

$$v_1(x) = -\frac{1}{2\omega_n} \int\limits_{\partial\Omega} \frac{1 - |x|^2}{|x - y|^n} \left[\varphi_2(y) - \frac{\partial v_0(y)}{\partial \nu}\right] dS_y.$$

Then, using the property

$$\int\limits_{\partial\Omega} \frac{\partial v_0(y)}{\partial \nu} dS_y = 0$$

of harmonic functions, we have

$$v(0) = v_0(0) + v_1(0) = \frac{1}{\omega_n} \int\limits_{\partial\Omega} \varphi_1(y) dS_y - \frac{1}{2\omega_n} \int\limits_{\partial\Omega} \left[\varphi_2(y) - \frac{\partial v_0(y)}{\partial \nu}\right] dS_y =$$

$$\frac{1}{2\omega_n} \int\limits_{\partial\Omega} \left[2\varphi_1(y) - \varphi_2(y) + \frac{\partial v_0(y)}{\partial \nu}\right] dS_y = \frac{1}{2\omega_n} \int\limits_{\partial\Omega} [2\varphi_1(y) - \varphi_2(y)]\, dS_y.$$

Thus equality (3.5) is proved.

Now we turn to the proof of (3.6).

A direct calculation gives

$$\frac{\partial v_0(0)}{\partial x_j} = \frac{1}{\omega_n} \int\limits_{\partial\Omega} \left[\frac{-2x_j}{|x - y|^n} - \frac{n}{2} \frac{1 - |x|^2}{|x - y|^{n+2}} \cdot 2(x_j - y_j)\right]\Bigg|_{x=0} \varphi_1(y) dS_y =$$

$$\frac{n}{\omega_n}\int\limits_{\partial\Omega} y_j\varphi_1(y)dS_y,$$

$$\frac{\partial v_1(0)}{\partial x_j} = -\frac{1}{2\omega_n}\int\limits_{\partial\Omega}\left[\frac{-2x_j}{|x-y|^n} - \frac{n}{2}\frac{1-|x|^2}{|x-y|^{n+2}}\cdot 2(x_j-y_j)\right]\Bigg|_{x=0}\times$$
$$\left[\varphi_2(y) - \frac{\partial v_0(y)}{\partial\nu}\right]dS_y = -\frac{n}{2\omega_n}\int\limits_{\partial\Omega} y_j\left[\varphi_2(y) - \frac{\partial v_0(y)}{\partial\nu}\right]dS_y.$$

If we denote $z_1(y) = y_j$, $z_2(y) = v_0(x)$ and note that these are harmonic functions, then by the Green formula we have

$$\int\limits_{\Omega}[z_1(y)\Delta z_2(y) - z_2(y)\Delta z_1(y)]\,dy = \int\limits_{\partial\Omega}\left[z_1(y)\frac{\partial z_2(y)}{\partial\nu} - \frac{\partial z_1(y)}{\partial\nu}z_2(y)\right]dS_y. \qquad (3.10)$$

Now we substitute the equality

$$\frac{\partial z_1(y)}{\partial\nu}\bigg|_{\partial\Omega} = \rho\frac{\partial y_j}{\partial\rho}\bigg|_{\partial\Omega} = \sum_{i=1}^{n} y_i\frac{\partial y_j}{\partial y_i}\bigg|_{\partial\Omega} = y_j|_{\partial\Omega}$$

to the Green formula (3.10). Then

$$0 = \int\limits_{\partial\Omega}\left[y_j\frac{\partial v_0(y)}{\partial\nu} - y_j v_0(y)\right]dS_y = \int\limits_{\partial\Omega}\left[y_j\frac{\partial v_0(y)}{\partial\nu} - y_j\varphi_1(y)\right]dS_y,$$

or

$$\int\limits_{\partial\Omega} y_j\frac{\partial v_0(y)}{\partial\nu}dS_y = \int\limits_{\partial\Omega} y_j\varphi_1(y)dS_y.$$

Therefore

$$\frac{\partial v_1(0)}{\partial x_j} = -\frac{n}{2\omega_n}\int\limits_{\partial\Omega}\left[y_j\varphi_2(y) - y_j\frac{\partial v_0(y)}{\partial\nu}\right]dS_y = -\frac{n}{2\omega_n}\int\limits_{\partial\Omega} y_j\left[\varphi_2(y) - \varphi_1(y)\right]dS_y.$$

Since

$$\frac{\partial v(0)}{\partial x_j} = \frac{\partial v_0(0)}{\partial x_j} + \frac{\partial v_1(0)}{\partial x_j},$$

we finally have

$$\frac{\partial v(0)}{\partial x_j} = \frac{n}{\omega_n}\int\limits_{\partial\Omega} y_j\varphi_1(y)dS_y - \frac{n}{2\omega_n}\int\limits_{\partial\Omega} y_j\left[\varphi_2(y) - \varphi_1(y)\right]dS_y =$$
$$\frac{n}{2\omega_n}\int\limits_{\partial\Omega} y_j\left[3\varphi_1(y) - \varphi_2(y)\right]dS_y.$$

□

**Lemma 3.4.** *Let $g(x), \varphi_1(x)$ and $\varphi_2(x)$ be sufficiently smooth functions and $g_1(x) = \Gamma_4[\Gamma_3[g]](x)$. Let $v(x)$ be the solution of problem (3.1). Then*

*1) $v(0) = 0$ if and only if*

$$\frac{1}{2}\int\limits_{\Omega}\left(1-|y|^2\right)\Gamma_3[g(y)]dy + \int\limits_{\partial\Omega}\left[2\varphi_1(y) - \varphi_2(y)\right]dS_y = 0; \qquad (3.11)$$

*2)* $\frac{\partial v(0)}{\partial x_j} = 0$ *for all* $j = 1, 2, ..., n,$ *if and only if*

$$\frac{1}{2}\int\limits_{\Omega} y_j(1-|y|^2)\Gamma_4\left[g\right](y)dy + \int\limits_{\partial\Omega} y_j\left[3\varphi_1(y) - \varphi_2(y)\right]dS_y = 0, j = 1, 2, ..., n. \tag{3.12}$$

*Proof.* We represent the solution of problem (3.1) in the form

$$v(x) = \int_{\Omega} G_{2,n}\left(x,y\right) g_1\left(y\right) dy + v_0(x) + (1-|x|^2)v_1(x),$$

where $v_0(x)$ and $v_1(x)$ are the solutions of problems (3.8) and (3.9) correspondingly.

If we make use of (3.3) and (3.5), then we obtain

$$v(0) = \int\limits_{\Omega} G_{2,n}\left(0,y\right) g_1\left(y\right) dy + v_0(0) + v_1(0) =$$

$$\frac{1}{4\omega_n}\int\limits_{\Omega}\left(1-|y|^2\right)\Gamma_3[g(y)]dy + \frac{1}{2\omega_n}\int\limits_{\partial\Omega}\left[2\varphi_1(y) - \varphi_2(y)\right]dS_y = 0.$$

Thus (3.11) is proved.

Similarly, making use of (3.4) and (3.6) we have

$$\frac{\partial v(0)}{\partial x_j} = \int\limits_{\Omega}\frac{\partial}{\partial x_j}G_{2,n}\left(0,y\right) g_1\left(y\right) dy + \frac{\partial v_0(0)}{\partial x_j} + \frac{\partial v_1(0)}{\partial x_j} =$$

$$\frac{n}{4\omega_n}\int\limits_{\Omega} y_j(1-|y|^2)\Gamma_4\left[g\right](y)dy + \frac{n}{2\omega_n}\int\limits_{\partial\Omega} y_j\left[3\varphi_1(y) - \varphi_2(y)\right]dS_y = 0,$$

$j = 1, 2, ..., n,$ which proves (3.12).

□

# 4 The Neumann Problem in the General Case

In this section we consider the Neumann problem (1.5), (1.6) and prove the following main result of the present paper.

**Theorem 4.1.** *Let* $\psi_k(x)$*,* $k = 1, 2$ *and* $g(x)$ *be sufficiently smooth. Then the necessary and sufficient solvability conditions for the Neumann boundary value problem (1.5), (1.6) have the form*

$$\frac{1}{2}\int\limits_{\Omega}\left(1-|y|^2\right)\Gamma_3[g(y)]dy = \int\limits_{\partial\Omega}\psi_2(y)dS_y, \tag{4.1}$$

$$\frac{1}{2}\int\limits_{\Omega} y_j(1-|y|^2)\Gamma_4\left[g\right](y)dy = \int\limits_{\partial\Omega} y_j\left[\psi_2(y) - \psi_1(y)\right]dS_y, j = 1, 2, ..., n \tag{4.2}$$

*If a solution exists, then it is unique up to a first order polynomial and can be represented as*

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j + \int\limits_0^1 (1-s)s^{-2}v(sx)ds, \tag{4.3}$$

*where* $c_j, j = 0, 1, ..., n,$ *are arbitrary constants,* $v(x)$ *is the solution of Dirichlet problem (3.1) with the functions* $g_1(x) = \left(r\frac{\partial}{\partial r} + 4\right)\left(r\frac{\partial}{\partial r} + 3\right)g(x)$*,* $\varphi_1(x) = \psi_1(x)$*,* $\varphi_2(x) = \psi_2(x) + 2\psi_1(x)$ *and with the additional conditions*

$$v(0) = 0, \ \ \frac{\partial v(0)}{\partial x_j} = 0, \ \ j = 1, 2, ..., n. \tag{4.4}$$

*Proof.* Let the solution of problem (1.5), (1.6) exist and let us denote it by $u(x)$. We prove, that

1) conditions (4.1) and (4.2) are satisfied;

2) the solution $u(x)$ has the form (4.3) and function $v(x)$ has the properties (4.4).

We apply the operator $r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)$ to function $u(x)$ and denote $v(x) = r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)u(x)$.

Now we define an equation and boundary conditions for the function $v(x)$. By virtue of equality (2.6) we have

$$\Delta^2 v(x) = \left(r\frac{\partial}{\partial r} + 4\right)\left(r\frac{\partial}{\partial r} + 3\right)\Delta^2 u(x) =$$
$$\left(r\frac{\partial}{\partial r} + 4\right)\left(r\frac{\partial}{\partial r} + 3\right)g(x) \equiv g_1(x).$$

It is not hard to verify, that for any $k = 1, 2, ...$ and all $x \in \partial\Omega$ one has [13]

$$\frac{\partial^k u(x)}{\partial \nu^k} = r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)...\left(r\frac{\partial}{\partial r} - (k-1)\right)u(x).$$

Therefore

$$v(x)|_{\partial\Omega} = \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)u(x)\Big|_{\partial\Omega} = \frac{\partial^2 u(x)}{\partial \nu^2}\Big|_{\partial\Omega} = \psi_1(x),$$

$$\frac{\partial v(x)}{\partial \nu}\Big|_{\partial\Omega} = r\frac{\partial v(x)}{\partial r}\Big|_{\partial\Omega} = r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)\right]u(x)\Big|_{\partial\Omega} =$$
$$r^3\frac{\partial^3 u(x)}{\partial r^3} + 2r^2\frac{\partial^2 u(x)}{\partial r^2}\Big|_{\partial\Omega} =$$
$$\frac{\partial^3 u(x)}{\partial \nu^3} + 2\frac{\partial^2 u(x)}{\partial \nu^2}\Big|_{\partial\Omega} = \psi_2(x) + 2\psi_1(x).$$

Thus, if $u(x)$ is a solution of problem (1.5), (1.6), then the function $v(x) = r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)u(x)$ will be a solution of the Dirichlet problem (3.1) with the functions

$$g_1(x) = \left(r\frac{\partial}{\partial r} + 4\right)\left(r\frac{\partial}{\partial r} + 3\right)g(x), \tag{4.5}$$

and

$$\varphi_1(x) = \psi_1(x), \ \ \varphi_2(x) = \psi_2(x) + 2\psi_1(x). \tag{4.6}$$

Moreover, a direct calculations show, that the function $v(x)$ satisfies the conditions (4.4).

According to Lemma 3.4, the conditions (4.4) are fulfilled if and only if the conditions (3.11) and (3.12) are satisfied. Since $\varphi_1(x) = \psi_1(x)$ and $\varphi_2(x) = \psi_2(x) + 2\psi_1(x)$, one has

$$\int_{\partial\Omega}[2\varphi_1(y) - \varphi_2(y)]\,dS_y = -\int_{\partial\Omega}\psi_2(y)dS_y,$$

$$\int_{\partial\Omega}y_j\,[3\varphi_1(y) - \varphi_2(y)]\,dS_y = \int_{\partial\Omega}y_j\,[\psi_1(y) - \psi_2(y)]\,dS_y, j = 1, 2, ..., n.$$

Thus the statement 1) is proved.

Now consider the integral

$$J = \int_0^1 (1-s)\frac{d^2 u(sx)}{ds^2}ds.$$

We can rewrite it as

$$J = \int_0^1 (1-s)s^{-2}s^2 \frac{d^2u(sx)}{ds^2}ds = \int_0^1 (1-s)s^{-2}s\frac{d}{ds}\left(s\frac{d}{ds}-1\right)u(sx)ds.$$

Integrating by parts gives

$$J = \int_0^1 (1-s)\frac{d^2u(sx)}{ds^2}ds = (1-s)\frac{du(sx)}{ds}\bigg|_{s=0}^{s=1} + \int_0^1 \frac{du(sx)}{ds}ds =$$

$$-\frac{du(0)}{ds} + u(x) - u(0).$$

Due to equality $s\frac{d}{ds}\left(s\frac{d}{ds}-1\right)u(sx) = v(sx)$ one has

$$u(x) = u(0) + \frac{du(0)}{ds} + \int_0^1 (1-s)s^{-2}v(sx)ds. \tag{4.7}$$

If we denote $y_j = sx_j$, $j = 1, 2, ..., n$, then

$$\frac{du(sx)}{ds} = \sum_{j=1}^n \frac{\partial u}{\partial y_j}\frac{dy_j}{ds} = \sum_{j=1}^n x_j\frac{\partial u}{\partial y_j},$$

or

$$\frac{du(0)}{ds} = \sum_{j=1}^n x_j\frac{\partial u(0)}{\partial y_j}. \tag{4.8}$$

Denoting $u(0) = c_0$, $\frac{\partial u(0)}{\partial y_j} = c_j$ and substituting (4.8) into (4.7), we obtain the statement 2).

Now we suppose that the statements 1) and 2) are satisfied and prove the existence of the solution of problem (1.5), (1.6).

Let $v(x)$ be the solution of the problem (3.1) with functions (4.5) and (4.6). From the statement 1) it follows that $v(x)$ satisfies the conditions (4.4). Therefore the following function

$$u(x) = c_0 + \sum_{j=1}^n c_jx_j + \int_0^1 (1-s)s^{-2}v(sx)ds$$

is well defined.

Next we prove that this function is a solution of problem (1.5), (1.6).

Apply operator $\Delta^2$ to $u(x)$. Then

$$\Delta^2u(x) = \int_0^1 (1-s)s^2g_1(sx)ds =$$

$$\int_0^1 (1-s)s^2\left(s\frac{d}{ds}+4\right)\left(s\frac{d}{ds}+3\right)g(sx)ds = \int_0^1 (1-s)s^2\Gamma_4\left[\Gamma_3\left[g\right]\right](sx)ds.$$

On the other hand

$$\Gamma_4^{-1}\left[\Gamma_3^{-1}\left[g_1\right]\right](x) = \int_0^1 t_1^3\int_0^1 t_2^2g_1(t_1t_2x)dt_2dt_1.$$

By changing variables $s = t_1 t_2$ one has

$$\Gamma_4^{-1}\left[\Gamma_3^{-1}[g_1]\right](x) = \int_0^1 \int_0^{t_1} s^2 g_1(sx)dsdt_1 = \int_0^1 s^2 g_1(sx) \int_s^1 dt_1 ds =$$

$$\int_0^1 (1-s)s^2 g_1(sx)ds.$$

Thus

$$\Delta^2 u(x) = \Gamma_4^{-1}\left[\Gamma_3^{-1}[g_1]\right](x) = \Gamma_4^{-1}\left[\Gamma_3^{-1}[\Gamma_3[\Gamma_4[g]]]\right](x),$$

and according to (2.3) we have $\Delta^2 u(x) = g(x)$.

Now we only have to prove that $u(x)$ satisfies the boundary condition (1.6).

By changing variables $sr = \xi, ds = r^{-1}d\xi$ we obtain

$$\int_0^1 (1-s)s^2 v(sx)ds = \int_0^r \left(1-\frac{\xi}{r}\right) r^2\xi^{-2}v(\xi\theta)r^{-1}d\xi = \int_0^r (r-\xi)\,\xi^{-2}v(\xi\theta)d\xi.$$

Therefore, using the equality $r\frac{\partial u(x)}{\partial r} = \sum_{k=1}^n x_k \frac{\partial u(x)}{\partial x_k}$, one has

$$r\frac{\partial u(x)}{\partial r} = \sum_{j=1}^n c_j x_j + r\int_0^r \xi^{-2}v(sx)ds,$$

$$\left(r\frac{\partial}{\partial r}-1\right) r\frac{\partial}{\partial r}u(x) = \left(r\frac{\partial}{\partial r}-1\right)\left[\sum_{j=1}^n c_j x_j + r\int_0^r \xi^{-2}v(sx)ds\right] =$$

$$\sum_{j=1}^n c_j x_j + r\int_0^r \xi^{-2}v(sx)ds + r^2\cdot r^{-2}v(x) - \sum_{j=1}^n c_j x_j - r\int_0^r \xi^{-2}v(sx)ds = v(x).$$

Then

$$\left.\frac{\partial^2 u(x)}{\partial \nu^2}\right|_{\partial\Omega} = \left(r\frac{\partial}{\partial r}-1\right) r\frac{\partial}{\partial r}u(x)\bigg|_{\partial\Omega} = v(x)|_{\partial\Omega} = \varphi_1(x) \equiv \psi_1(x).$$

In the same way

$$\left(r\frac{\partial}{\partial r}-2\right)\left(r\frac{\partial}{\partial r}-1\right) r\frac{\partial}{\partial r}u(x) = \left(r\frac{\partial}{\partial r}-2\right)v(x) = r\frac{\partial v(x)}{\partial r} - 2v(x).$$

Hence

$$\left.\frac{\partial^3 u(x)}{\partial \nu^3}\right|_{\partial\Omega} = \left.\frac{\partial v(x)}{\partial \nu} - 2v(x)\right|_{\partial\Omega} = \varphi_2(x) - 2\varphi_1(x) = \psi_2(x),$$

i.e. function $u(x)$ satisfies the boundary conditions (1.6). □

**Example 4.2.** *Consider the follwing Neumann problem*

$$\Delta^2 u(x) = 1, \quad x \in \Omega$$

$$\frac{\partial^2 u(x)}{\partial \nu^2} = a, \quad x \in \partial\Omega,$$

$$\frac{\partial^3 u(x)}{\partial \nu^3} = b, \quad x \in \partial\Omega,$$

*where $a$ and $b$ are given constants. Applying Theorem 4.1 we will find solvability conditions and the solution of this problem.*

We have $g(x) = 1$, $\psi_1(x) = a$ and $\psi_2(x) = b$. Therefore $\Gamma_3[g](x) = 3$, $\Gamma_4[g](x) = 4$ and $\Gamma_3[\Gamma_4[g]](x) = 12$.

We first check whether or not the boundary functions of the considering problem satisfy the conditions (4.1) and (4.2). It is not hard to verify, that

$$\int_{\partial\Omega} \psi_2(y)dS_y = b \cdot \omega_n, \ \int_{\Omega} \frac{1-|y|^2}{2}\Gamma_3[g](y)dy = \frac{3\omega_n}{n(n+2)}.$$

Therefore condition (4.1) has the form $b = \frac{3}{n(n+2)}$.

Since the function $y_j$ is odd, then one obviously has

$$\int_{\Omega} y_j \frac{1-|y|^2}{2}\Gamma_4[g](y)dy = 0$$

and

$$\int_{\partial\Omega} y_j(b-2a)dS_y = 0.$$

Hence conditions (4.2) are satisfied for any $j = 1, 2..., n$.

Thus the considering Neumann problem has a solution if and only if

$$b = \frac{3}{n(n+2)}.$$

Noting this we may write the corresponding Dirichlet problem in the form

$$\Delta^2 v(x) = 12, \quad x \in \Omega,$$

$$v(x) = a \ \text{ and } \ \frac{\partial v(x)}{\partial \nu} = 2a + \frac{3}{n(n+2)}, \quad x \in \partial\Omega.$$

The solution of this problem can be written as (see [14]):

$$v(x) = \big(2a - \frac{3}{n(n+2)}\big)|x|^2 + \frac{3}{n(n+2)}|x|^4.$$

It is easy to verify that this function satisfies conditions (4.4).

According to formula (4.3), after some routine calculation, we obtain the solution of the considering Neumann problem in the form

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j + \big(\frac{a}{2} - \frac{3}{n(n+2)}\big)|x|^2 + \frac{1}{8n(n+2)}|x|^4.$$

# 5 A Different Formulation of the Main Result

If we integrate by part in the integral (3.3) one more time, then we obtain

$$v(0) = \frac{1}{\omega_n} \cdot \frac{n-1}{4} \int_{\Omega} |y|^2 g(y)dy - \frac{1}{\omega_n} \cdot \frac{n-3}{4} \int_{\Omega} g(y)dy.$$

Integration one more time in the integral (3.4) gives

$$\frac{\partial v(0)}{\partial x_j} = \frac{1}{\omega_n} \cdot \frac{n(n-1)}{4} \int_{\Omega} y_j |y|^2 g(y)dy - \frac{1}{\omega_n} \cdot \frac{n(n-3)}{4} \int_{\Omega} y_j g(y)dy,$$

$j = 1, 2, ..., n$.

Having this done, if we repeat the other part of the proof of Theorem 4.1, then we have the following formulation of the main result.

**Theorem 5.1.** *Let $\psi_k(x)$, $k = 1, 2$ and $g(x)$ be sufficiently smooth. Then the necessary and sufficient solvability condition for the Neumann boundary value problem (1.5), (1.6) has the form*

$$\frac{n-1}{2} \int_\Omega |y|^2 g(y)dy - \frac{n-3}{2} \int_\Omega g(y)dy = \int\limits_{\partial\Omega} \psi_2(x)dS_y,$$

$$\frac{n-1}{2} \int_\Omega y_j |y|^2 g(y)dy - \frac{n-3}{2} \int_\Omega y_j g(y)dy =$$

$$\int\limits_{\partial\Omega} y_j \left[\psi_2(y) - \psi_1(y)\right] dS_y, \ \ j = 1, 2, ..., n$$

*If a solution exists, then it is unique up to a first order polynomial and can be represented as*

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j + \int\limits_0^1 (1-s)s^{-2} v(sx)ds,$$

*where $c_j, j = 0, 1, ..., n,$ are arbitrary constants, $v(x)$ is the solution of Dirichlet problem (3.1) with the functions $g_1(x) = \left(r\frac{\partial}{\partial r} + 4\right)\left(r\frac{\partial}{\partial r} + 3\right) g(x)$, $\varphi_1(x) = \psi_1(x)$, $\varphi_2(x) = \psi_2(x) + 2\psi_1(x)$ and with the additional conditions*

$$v(0) = 0, \ \ \frac{\partial v(0)}{\partial x_j} = 0, \ \ j = 1, 2, ..., n.$$

# 6 Conclusions

**a** In the section two we study properties of some integro-differential operators, which we then use throughout the paper.

**b** In section 3 we investigate the Dirichlet problem for biharmonic equation, making use of the explicit form of the Green function found in [9, 10, 11].

**c** Then in the following section 4, reducing the Neumann problem (1.5), (1.6) to the considered Dirichlet problem, we give the necessary and sufficient solvability conditions for the Neumann problem for the biharmonic equation. This is the main result of the paper. In these conditions the right hand side of the equation $g(x)$ participates under the operators $\Gamma_3$ and $\Gamma_4$.

**d** Finally, in section 5, slightly modifying the proof of the main result, we will give it in a different formulation, where one does not have operators $\Gamma_3$ and $\Gamma_4$.

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# Competing Interests

The authors declare that no competing interests exist.

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