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Fractional Sub-equation Method and Analytical Solutions to the Hirota-satsuma Coupled KdV Equation and Coupled mKdV Equation

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Abstract

The fractional sub-equation method is proposed to construct analytical solutions of nonlinear fractional partial differential equations (FPDEs), involving Jumarie's modified Riemann-Liouville derivative. The fractional sub-equation method is applied to the space-time fractional generalized Hirota-Satsuma coupled KdV equation and coupled mKdV equation. The analytical solutions show that the fractional sub-equation method is very effective for the fractional coupled KdV and mKdV equations. The solutions are compared with that of the extended tanh-function method. New exact solutions are found for the coupled mKdV equation.

Keywords: Fractional sub-equation method; Analytical solutions; Nonlinear KdV and mKdV fractional equations.

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1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. In recent years, nonlinear fractional differential equations (FDEs) have gained considerable interest. It is caused by the development of the theory of fractional calculus itself but also by the applications of such constructions in various sciences such as physics, engineering, biology and others areas [1–7]. Among the investigations for fractional differential equations, research for seeking exact solutions is an important topic as well as applying them to practical problems [8–13]. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of FDEs. For example, the finite difference method [14], the finite element method [15, 16], the differential transform method [17,18], the adomian decomposition method (ADM) [19–21], the variational iteration method [22–24], the homotopy perturbation method [25], the Jacobi elliptic-function method, the modified trigonometric function series method, the modified (G'/G) expansion method and other methods, have been applied to construct analytical traveling

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wave solutions for the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity [26–30]. Several important aspects of FPDEs have been investigated in recent years; such as the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical solutions, and the stability of solutions [31,32].

By taking into account the results obtained by [33], Zhang and Zhang [34] proposed a new algebraic method named the fractional sub-equation method to look for traveling wave solutions of nonlinear FPDEs. The method is based on the homogeneous balance principle [35] and the Jumarie's modified Riemann-Liouville derivative of fractional order [36, 37]. With the help of this method, Zhang et al. successfully obtained traveling wave solutions of nonlinear time fractional biological population model and (4+1)-dimensional space-time fractional Fokas equation [34]. Also the validity and advantages of the method have been applied to the space-time fractional mBBM equation and the space-time fractional ZKBBM equation [38]. The sub-equation method also has been applied to the space–time fractional potential Kadomtsev–Petviashvili (PKP) equation and the space–time fractional symmetric regularized long wave (SRLW) equation [39]. The coupled KdV and mKdV equations, which were introduced by Wu et al. [40], have been analyzed applying the sub-equation method in the integer order limit case, called the extended tanh-function method [41–44]. Also the improved fractional sub-equation method has been applied to solve the nonlinear FPDE KdV coupled equation in fluid mechanics [45].

The coupled KdV and mKdV systems are very complicated and not easy to solve by direct integration method. Fan [41] has provided a suggestion to construct soliton solutions for these equations by using an extended tanh-function method and symbolic computation.

The aim of this work is to obtain analytical solutions, by applying the sub-equation method [34], for the space-time fractional Hirota-Satsuma coupled KdV equation,

$$\begin{aligned} D_t^{\alpha}u &= \frac{1}{2}D_x^{3\alpha}u - 3uD_x^{\alpha}u + 3D_x^{\alpha}\left(vw\right) \quad , \\ D_t^{\alpha}v &= -D_x^{3\alpha}v + 3uD_x^{\alpha}v \, , \\ D_t^{\alpha}w &= -D_x^{3\alpha}w + 3uD_x^{\alpha}w \ , \ t>0 \, , 0<\alpha \le 1, \end{aligned} \tag{1}$$

and the space-time fractional coupled mKdV equation,

$$\begin{aligned} D_t^{\alpha}u &= \frac{1}{2}D_x^{3\alpha}u - 3u^2D_x^{\alpha}u + \frac{3}{2}D_x^{2\alpha}v + 3D_x^{\alpha}\left(uv\right) - 3\lambda D_x^{\alpha}u \quad , \\ D_t^{\alpha}v &= -D_x^{3\alpha}v - 3vD_x^{\alpha}v - 3\left(D_x^{\alpha}u\right)\left(D_x^{\alpha}v\right) + 3u^2D_x^{\alpha}v + 3\lambda D_x^{\alpha}v \quad , \\ & t>0 \ , \ 0<\alpha \le 1 \, , \end{aligned} \tag{2}$$

where D^{α}_{x} and D^{α}_{t} are the Jumarie's modified Riemann-Liouville derivatives. λ is a constant and α is the parameter describing the order of the fractional derivatives of $u(x,t)$, $v(x,t)$ and $w(x,t)$. The obtained solutions would be important for previous works where approximated methods [46–49] have been applied to solve the coupled equations (1) and (2). The Hirota-Satsuma coupled KdV equation describes the interaction between two long waves with different dispersion relations. It is

a non-linear equation that exhibits special solutions, known as solitons, which are stable and do not disperse with time [50]. In Ref. [40] the authors have been introduced a 4x4 matrix spectral problem with three potentials, by which the coupled mKdV equation was obtained as a new integrable generalization of the Hirota-Satsuma coupled KdV equation.

The outline of this work is as follows: in section 2, the sub-equation method is presented. Section 3 contains the application of the method to solve the coupled KdV and mKdV FPDEs. In section 4, we discuss the reliability of the proposed method and the exact solutions are compared with the results reported in the literature [41]. Finally in section 5 some conclusions are presented.

2. Description of the Fractional Sub-equation Method and Its Applications to the Space-Time Fractional Differential Equations

In this section we present the main ideas of the fractional sub-equation method. This method considers the Jumarie's modified Riemann-Liouville fractional derivative of order α [36,37]:

$$D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} \left[f(\xi) - f(0) \right] d\xi, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} \left[f(\xi) - f(0) \right] d\xi & 0 < \alpha < 1 \\ \left[f^{(\alpha-n)}(x) \right]^{(n)} & n \le \alpha \le n+1,\ n \ge 1. \end{cases} \tag{3}$$

Some properties for the proposed modified Riemann–Liouville derivative are [36,37]:

$$\begin{aligned} D_x^{\alpha} x^{\gamma} &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha} \\ D_x^{\alpha}\, c &= 0, \quad \alpha \ge 0 \quad c = const. \\ D_x^{\alpha} \left(c f(x) \right) &= c \left(D_x^{\alpha} f(x) \right), \quad \alpha \ge 0 \quad c = const. \\ D_x^{\alpha} \left(f(x) g(x) \right) &= g(x) \left(D_x^{\alpha} f(x) \right) + f(x) \left(D_x^{\alpha} g(x) \right) \\ D_x^{\alpha} f\left[g(x) \right] &= f_g'\left[g(x) \right] D_x^{\alpha} g(x) = \left(D_g^{\alpha} f\left[g(x) \right] \right) \left(g'(x) \right)^{\alpha} . \end{aligned} \tag{4}$$

The above properties play an important role in the fractional sub-equation method. The main steps of this method are described as follows [34,38,39]:

Step 1: Suppose that a nonlinear FPDE, say in two independent variables, is given by:

$$P\left(u, u_x, u_t, D_x^{\alpha} u, D_t^{\alpha} u, \ldots \right) = 0 \quad 0 < \alpha \le 1 \ , \tag{5}$$

where $D^{\alpha}_{x} u$ and $D^{\alpha}_{t} u$ are the Jumarie's modified Riemann-Liouville derivatives of u, $u = u(x,t)$ is

an unknown function, P is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 2: By using the traveling wave transformation

$$u(x,t) = u(\xi), \quad \xi = kx + ct, \tag{6}$$

where k and c are constants to be determined later, the FPDE (5) is reduced to the following nonlinear fractional ordinary differential equation for $u(x,t) = u(\xi)$:

$$P\left(u, cu', ku', k^{\alpha} D_{\xi}^{\alpha} u, c^{\alpha} D_{\xi}^{\alpha} u, \ldots\right) = 0 \quad . \tag{7}$$

Step 3: We suppose that Eq. (7) has the following solution:

$$u(\xi) = \sum_{i=0}^{n} a_i \phi^i \quad , \tag{8}$$

where a_i (i=0,1,2,…,n) are constants to be determined later, n is a positive integer determined by balancing the highest order derivatives and nonlinear terms in Eq. (7), and $\phi = \phi(\xi)$ satisfies the following fractional Riccati equation:

$$D_{\xi}^{\alpha} \phi = \sigma + \phi^2 \quad , \tag{9}$$

where σ is a constant. By using the generalized exp-function method via Mittag-Leffler function, Zhang et al. [33], obtained the following solutions of fractional Riccati equation (9):

$$\phi(\xi) = \begin{cases} -\sqrt{-\sigma}\,\tanh_{\alpha}\left(\sqrt{-\sigma}\xi\right) & \sigma < 0 \\ -\sqrt{-\sigma}\,\coth_{\alpha}\left(\sqrt{-\sigma}\xi\right) & \sigma < 0 \\ \sqrt{\sigma}\,\tan_{\alpha}\left(\sqrt{\sigma}\xi\right) & \sigma > 0 \\ \sqrt{\sigma}\,\cot_{\alpha}\left(\sqrt{\sigma}\xi\right) & \sigma > 0 \\ -\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+\omega}, \quad \omega = const. & \sigma = 0\,, \end{cases} \tag{10}$$

where the generalized hyperbolic and trigonometric functions are defined as:

$$\begin{aligned}
&\sinh_\alpha(x)=\frac{E_\alpha(x^\alpha)-E_\alpha(-x^\alpha)}{2},\quad \cosh_\alpha(x)=\frac{E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)}{2},\\
&\tanh_\alpha(x)=\frac{\sinh_\alpha(x)}{\cosh_\alpha(x)},\qquad \coth_\alpha(x)=\frac{\cosh_\alpha(x)}{\sinh_\alpha(x)},\\
&\sin_\alpha(x)=\frac{E_\alpha(ix^\alpha)-E_\alpha(-ix^\alpha)}{2},\quad \cos_\alpha(x)=\frac{E_\alpha(ix^\alpha)+E_\alpha(-ix^\alpha)}{2},\\
&\tan_\alpha(x)=\frac{\sin_\alpha(x)}{\cos_\alpha(x)},\qquad \cot_\alpha(x)=\frac{\cos_\alpha(x)}{\sin_\alpha(x)},
\end{aligned} \tag{11}$$

where $E_\alpha(z)$ is the Mittag-Leffler function, given as:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}. \tag{12}$$

Step 4: Substituting Eq. (8) into Eq. (7) taking into account Eq. (9) and the properties of the Jumarie's modified Riemann-Liouville derivative, Eq. (7) is converted to a polynomial in $\phi^i(\xi)$ (i=0,1,2,…). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for a_i (i=0,1,2,…), k and c.

Step 5: Solving the equations system in Step 4, and by using the solutions Eq. (10), we can construct a variety of exact solutions of Eq. (7).

Remark: If $\alpha\rightarrow 1$, the Riccati equation becomes $\phi'(\xi)=\sigma+\phi^2(\xi)$. The method can be used to solve integer order differential equations. In this sense the sub-equation method includes the existing tanh-function method as special case.

3. Fractional Sub-equation Method Applied to the Kdv and MKDV Coupled Equations

In this section we apply the fractional sub-equation method to construct the exact solutions for space-time fractional Hirota-Satsuma coupled KdV equation (1) and the coupled mKdV equation (2).

3.1 The Nonlinear Hirota-Satsuma Coupled KdV Equation

To look for the travelling wave solution of Eq. (1), we make the transformation $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$, $w(x,t)=w(\xi)$, where $\xi=kx+ct$ and Eq. (1) can be reduced to the following nonlinear fractional ordinary differential equations (ODEs):

$$
\begin{aligned}
c^{\alpha} D_{\xi}^{\alpha} u &= \frac{k^{3\alpha}}{2} D_{\xi}^{3\alpha} u - 3k^{\alpha} u D_{\xi}^{\alpha} u + 3k^{\alpha} D_{\xi}^{\alpha} (vw) \\
c^{\alpha} D_{\xi}^{\alpha} v &= -k^{3\alpha} D_{\xi}^{3\alpha} v + 3k^{\alpha} u D_{\xi}^{\alpha} v \\
c^{\alpha} D_{\xi}^{\alpha} w &= -k^{3\alpha} D_{\xi}^{3\alpha} w + 3k^{\alpha} u D_{\xi}^{\alpha} w \quad .
\end{aligned} \tag{13}
$$

We suppose that Eq. (13) has the following formal solution:

$$
u(\xi) = \sum_{i=0}^{m_{\max}} u_i \phi^i \,,\; v(\xi) = \sum_{j=0}^{n_{\max}} v_j \phi^j \,,\; w(\xi) = \sum_{k=0}^{l_{\max}} w_k \phi^k \,, \tag{14}
$$

where $\phi(\xi)$ satisfies Eq. (9). Balancing the highest order linear terms and nonlinear terms in Eq. (13), it is possible to determine the value of $m_{\max}$, $n_{\max}$ and $l_{\max}$. Putting together Eq. (14) along with Eq. (9) into Eq. (13), we obtain the following two ansätze [41]:

$$
\begin{aligned}
u(\xi) &= u_0 + u_1\phi + u_2\phi^2 \quad , \\
v(\xi) &= v_0 + v_1\phi + v_2\phi^2 \quad , \\
w(\xi) &= w_0 + w_1\phi + w_2\phi^2 \quad ,
\end{aligned} \tag{15}
$$

where $m_{\max}= n_{\max}= l_{\max} =2$ and the second ansätz is given by:

$$
\begin{aligned}
u(\xi) &= u_0 + u_1\phi + u_2\phi^2 \quad , \\
v(\xi) &= v_0 + v_1\phi \quad , \\
w(\xi) &= w_0 + w_1\phi \quad ,
\end{aligned} \tag{16}
$$

with $m_{\max} = 2$, $n_{\max} = l_{\max} = 1$. Now for the case where $m_{\max}= n_{\max}= l_{\max} =2$, substituting Eq. (15) into Eq. (13) we obtain the following algebraic system equations:

$$
\begin{aligned}
c^{\alpha}(u_1 + 2u_2\phi)(\sigma + \phi^2) &= k^{3\alpha}(\sigma + \phi^2)(u_1\sigma + 8\sigma u_2\phi + 3u_1\phi^2 + 12u_2\phi^3) \\
&\quad -3k^{\alpha}(\sigma + \phi^2)(u_0 + u_1\phi + u_2\phi^2)(u_1 + 2u_2\phi) \\
&\quad +3k^{\alpha}(\sigma + \phi^2)(v_1 + 2v_2\phi)(w_0 + w_1\phi + w_2\phi^2) \\
&\quad +3k^{\alpha}(\sigma + \phi^2)(w_1 + 2w_2\phi)(v_0 + v_1\phi + v_2\phi^2)
\end{aligned} \tag{17}
$$

$$c^{\alpha}\left(v_1+2v_2\phi\right)\left(\sigma+\phi^2\right)=-k^{3\alpha}\left(\sigma+\phi^2\right)\left(2v_1\sigma+16\sigma v_2\phi+6v_1\phi^2+24v_2\phi^3\right)$$
$$+3k^{\alpha}\left(\sigma+\phi^2\right)\left(u_0+u_1\varphi+u_2\phi^2\right)\left(v_1+2v_2\phi\right)$$

(18)

$$c^{\alpha}\left(w_1+2w_2\phi\right)\left(\sigma+\phi^2\right)=-k^{3\alpha}\left(\sigma+\phi^2\right)\left(2w_1\sigma+16\sigma w_2\phi+6w_1\phi^2+24w_2\phi^3\right)$$
$$+3k^{\alpha}\left(\sigma+\phi^2\right)\left(u_0+u_1\varphi+u_2\phi^2\right)\left(w_1+2w_2\phi\right) \quad . \tag{19}$$

Now by setting the coefficients of ϕ^i (i=0,1,2,3) to zero, we obtain a set of algebraic equations for u_i, v_i, w_i (i=0,1,2), k and c:

$$\begin{aligned}
\phi^0 &: c^{\alpha}u_1=k^{3\alpha}\sigma u_1-3k^{\alpha}u_1u_0+3k^{\alpha}\left(v_1w_0+v_0w_1\right)\\
\phi^1 &: 2c^{\alpha}u_2=8k^{3\alpha}\sigma u_2-3k^{\alpha}\left(u_1^2+2u_0u_2\right)+6k^{\alpha}\left(v_2w_0+v_1w_1+v_0w_2\right)\\
\phi^2 &: 0=3k^{3\alpha}u_1-9k^{\alpha}u_1u_2+9k^{\alpha}\left(v_2w_1+v_1w_2\right)\\
\phi^3 &: 0=12k^{3\alpha}u_2-6k^{\alpha}u_2^2-12k^{\alpha}v_2w_2 \quad ,
\end{aligned} \tag{20}$$

obtained from equation Eq. (17),

$$\begin{aligned}
\phi^0 &: c^{\alpha}v_1=-2k^{3\alpha}\sigma v_1+3k^{\alpha}u_0v_1\\
\phi^1 &: 2c^{\alpha}v_2=-16k^{\alpha}\sigma v_2+k^{\alpha}\left(6u_0v_2+3u_1v_1\right)\\
\phi^2 &: 0=-6k^{3\alpha}v_1+3k^{\alpha}\left(2u_1v_2+u_2v_1\right)\\
\phi^3 &: 0=-24k^{3\alpha}v_2+6k^{\alpha}u_2v_2 \quad ,
\end{aligned} \tag{21}$$

obtained from equation Eq. (18),

$$\begin{aligned}
\phi^0 &: c^{\alpha}w_1=-2k^{3\alpha}\sigma w_1+3k^{\alpha}u_0w_1\\
\phi^1 &: 2c^{\alpha}w_2=-16k^{\alpha}\sigma w_2+k^{\alpha}\left(6u_0w_2+3u_1w_1\right)\\
\phi^2 &: 0=-6k^{3\alpha}w_1+3k^{\alpha}\left(2u_1w_2+u_2w_1\right)\\
\phi^3 &: 0=-24k^{3\alpha}w_2+6k^{\alpha}u_2w_2 \quad ,
\end{aligned} \tag{22}$$

obtained from equation Eq. (19). From these algebraic equations we notice that:

$$
\begin{aligned}
&u_1 = v_1 = w_1 = 0 \\
&u_2 = 4k^{2\alpha} \\
&v_2 = \frac{4k^{4\alpha}}{w_2} \\
&u_0 = \frac{c^\alpha + 8\sigma k^{3\alpha}}{3k^\alpha} \\
&v_0 = \frac{4\left(2k^\alpha c^\alpha w_2 + 4\sigma k^{4\alpha} w_2 - 3k^{4\alpha} w_0\right)}{3w_2^2} \\
&w_0 = w_2 = const.\ ,
\end{aligned}
\tag{23}
$$

and the tanh-type solution of the coupled space-time fractional KdV equation is given by:

$$
\begin{aligned}
&u(x,t) = \frac{c^\alpha + 8\sigma k^{3\alpha}}{3k^\alpha} + 4k^{2\alpha} \tanh_\alpha^2\left[\sqrt{-\sigma}\left(kx + ct\right)\right], \\
&v(x,t) = \frac{4\left(2k^\alpha c^\alpha w_2 + 4\sigma k^{4\alpha} w_2 - 3k^{4\alpha} w_0\right)}{3w_2^2} + \frac{4k^{4\alpha}}{w_2} \tanh_\alpha^2\left[\sqrt{-\sigma}\left(kx + ct\right)\right] \quad ,(24) \\
&w(x,t) = w_0 + w_2 \tanh_\alpha^2\left[\sqrt{-\sigma}\left(kx + ct\right)\right] \quad .
\end{aligned}
$$

Now if we consider the second ansätz (Eq. (16)), setting $v_2 = w_2 = 0$ into Eq. (20)-(22), the value of the parameters u_i (i=0,1,2), v_j , w_j (j=0,1), k and c, are given by:

$$
\begin{aligned}
&v_2 = w_2 = 0 \\
&u_1 = 0 \\
&u_2 = 2k^{2\alpha} \\
&u_0 = \frac{c^\alpha + 2\sigma k^{3\alpha}}{3k^\alpha} \\
&v_1 = \frac{4\left(k^\alpha c^\alpha - \sigma k^{4\alpha}\right)}{3w_1} \\
&v_0 = -\frac{4w_0\left(k^\alpha c^\alpha - \sigma k^{4\alpha}\right)}{3w_1^2} \\
&w_0 = w_1 = const.\ ,
\end{aligned}
\tag{25}
$$

and the tanh-type solution of the coupled space-time fractional KdV equation is given by:

$$u(x,t)=\frac{c^{\alpha}+2\sigma k^{3\alpha}}{3k^{\alpha}}+2k^{2\alpha}\tanh_{\alpha}^{2}\left[\sqrt{-\sigma}\left(kx+ct\right)\right] \ ,$$
$$v(x,t)=-\frac{4w_0\left(k^{\alpha}c^{\alpha}-\sigma k^{4\alpha}\right)}{3w_1^{2}}+\frac{4\left(k^{\alpha}c^{\alpha}-\sigma k^{4\alpha}\right)}{3w_1}\tanh_{\alpha}\left[\sqrt{-\sigma}\left(kx+ct\right)\right] \ , \quad (26)$$
$$w(x,t)=w_0+w_1\tanh_{\alpha}\left[\sqrt{-\sigma}\left(kx+ct\right)\right] \ .$$

We note that the solutions (24) and (26) in the limit case $\alpha\to1$ with σ =-1 are the same solutions previously obtained in Ref. [41].

3.2 The Nonlinear Coupled mKdV Equation

Now to look for the travelling wave solutions for the space-time fractional coupled mKdV equation, we make the transformation $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$ where $\xi=kx+ct$ and Eq. (2) can be reduced to the following nonlinear fractional ODEs:

$$c^{\alpha}D_{\xi}^{\alpha}u=\frac{k^{3\alpha}}{2}D_{\xi}^{3\alpha}u-3k^{\alpha}u^{2}D_{\xi}^{\alpha}u+\frac{3k^{2\alpha}}{2}D_{\xi}^{2\alpha}v+3k^{\alpha}D_{\xi}^{\alpha}\left(uv\right)-3\lambda k^{\alpha}D_{\xi}^{\alpha}u$$
$$c^{\alpha}D_{\xi}^{\alpha}v=-k^{3\alpha}D_{\xi}^{3\alpha}v-3k^{\alpha}vD_{\xi}^{\alpha}v-3k^{2\alpha}D_{\xi}^{\alpha}uD_{\xi}^{\alpha}v+3k^{\alpha}u^{2}D_{\xi}^{\alpha}v+3\lambda k^{\alpha}D_{\xi}^{\alpha}v \quad , \quad (27)$$

where

$$u\left(\xi\right)=\sum_{i=0}^{m_{\max}}u_i\phi^{i} \ , \ v\left(\xi\right)=\sum_{j=0}^{n_{\max}}v_j\phi^{j} \quad . \quad (28)$$

Balancing the highest order linear terms and nonlinear terms in Eq. (27) it is possible to determine the value of $m_{\max}$ and $n_{\max}$. Putting together Eq. (28) along with Eq. (9) into Eq. (27), we obtain the following two ansätze [41]:

$$u\left(\xi\right)=u_0+u_1\phi \ ,$$
$$v\left(\xi\right)=v_0+v_1\phi+v_2\phi^{2} \ , \quad (29)$$

with $m_{\max}$ =1, $n_{\max}$ =2 and

$$u\left(\xi\right)=u_0+u_1\phi \ ,$$
$$v\left(\xi\right)=v_0+v_1\phi \ , \quad (30)$$

with $m_{\max}$ =1, $n_{\max}$ =1. Now for the case where $m_{\max}$ =1 and $n_{\max}$ =2, substituting Eq. (29) into Eq. (27) we obtain the following algebraic system equations:

$$
\begin{aligned}
c^{\alpha}u_1\left(\sigma+\phi^2\right) &= k^{3\alpha}u_1\left(\sigma+\phi^2\right)\left(\sigma+3\phi^2\right)-3k^{\alpha}u_1\left(\sigma+\phi^2\right)\left(u_0+u_1\phi\right)^2 \\
&\quad +3k^{2\alpha}\left(\sigma+\phi^2\right)\left[v_1\phi+2v_2\phi^2+v_2\left(\sigma+\phi^2\right)\right] \\
&\quad +3k^{\alpha}\left(\sigma+\phi^2\right)\left[u_0v_1+u_1v_0+2\left(u_0v_2+u_1v_1\right)\phi+3u_1v_2\phi^2\right] \\
&\quad -3\lambda k^{\alpha}u_1\left(\sigma+\phi^2\right) \\
c^{\alpha}\left(v_1+2v_2\phi\right)\left(\sigma+\phi^2\right) &= -k^{3\alpha}\left(\sigma+\phi^2\right)\left(2v_1\sigma+16\sigma v_2\phi+6v_1\phi^2+24v_2\phi^3\right) \\
&\quad -3k^{\alpha}\left(\sigma+\phi^2\right)\left(v_1+2v_2\phi\right)\left(v_0+v_1\phi+v_2\phi^2\right) \\
&\quad -3k^{2\alpha}\left(\sigma+\phi^2\right)^2u_1\left(v_1+2v_2\phi\right) \\
&\quad +3k^{\alpha}\left(\sigma+\phi^2\right)\left(v_1+2v_2\phi\right)\left(u_0+u_1\phi\right)^2 \\
&\quad +3\lambda k^{\alpha}\left(\sigma+\phi^2\right)\left(v_1+2v_2\phi\right) \quad .
\end{aligned}
\tag{31}
\tag{32}
$$

Now by setting the coefficients of ϕ^i (i=0,1,2,3) to zero, we obtain a set of algebraic equations for u_i (i=0,1), v_j (j=0,1,2) , k and c:

$$
\begin{aligned}
\phi^0 &: \; c^{\alpha}u_1 = k^{3\alpha}\sigma u_1-3k^{\alpha}u_1u_0^2+3k^{\alpha}\left(u_1v_0+u_0v_1\right)-3\lambda k^{\alpha}u_1+3k^{2\alpha}\sigma v_2 \\
\phi^1 &: \; 0 = -6k^{\alpha}u_0u_1^2+3k^{2\alpha}v_1+6k^{\alpha}\left(u_1v_1+u_0v_2\right) \\
\phi^2 &: \; 0 = 3k^{3\alpha}u_1-3k^{\alpha}u_1^3+9k^{2\alpha}v_2+9k^{\alpha}u_1v_2 \quad ,
\end{aligned}
\tag{33}
$$

obtained from equation Eq. (31), and

$$
\begin{aligned}
\phi^0 &: \; c^{\alpha}v_1 = -2k^{3\alpha}\sigma v_1-3k^{\alpha}v_0v_1-3k^{2\alpha}\sigma u_1v_1+3k^{\alpha}u_0^2v_1+3\lambda k^{\alpha}v_1 \\
\phi^1 &: \; 2c^{\alpha}v_2 = -16k^{3\alpha}\sigma v_2-3k^{\alpha}\left(v_1^2+2v_0v_2\right)-6k^{2\alpha}\sigma u_1v_2 \\
&\quad +6k^{\alpha}\left(u_0^2v_2+u_0u_1v_1\right)+6\lambda k^{\alpha}v_2 \\
\phi^2 &: \; 0 = -6k^{3\alpha}v_1-9k^{\alpha}v_1v_2-3k^{2\alpha}u_1v_1+3k^{\alpha}u_1^2v_1+12k^{\alpha}u_0u_1v_2 \\
\phi^3 &: \; 0 = -24k^{3\alpha}v_2-6k^{\alpha}v_2^2+6k^{\alpha}u_1^2v_2-6k^{2\alpha}u_1v_2 \quad ,
\end{aligned}
\tag{34}
$$

obtained from equation Eq. (32). From these algebraic equations we notice that:

$$\begin{aligned} u_0 &= v_1 = 0 \quad , \\ u_1 &= -k^{\alpha} \quad , \\ v_2 &= -2k^{2\alpha} \quad , \\ v_0 &= \lambda - 2k^{2\alpha}\sigma \quad , \\ c^{\alpha} &= k^{3\alpha}\sigma \quad . \end{aligned} \tag{35}$$

As we know the equation (27) have the kink-type soliton solution:

$$u(x,t) = k^{\alpha}\tanh_{\alpha}(kx+ct) \; , \; v(x,t) = v_0 - 2k^{2\alpha}\tanh_{\alpha}^2(kx+ct) \quad , \tag{36}$$

constructed by Fan [41], for the ansätz proposed in Eq. (29). We can compare this solution with Eq. (10) for the tanh-type solution and obtain the following values for the coefficients u_i and v_j, the parameters k, c and the constant σ:

$$\begin{aligned} u_0 &= v_1 = 0 \quad , \\ u_1 &= -k^{\alpha} \quad , \\ v_2 &= -2k^{2\alpha} \quad , \\ v_0 &= \lambda + 2k^{2\alpha} \quad , \\ c^{\alpha} &= -k^{3\alpha} \quad , \\ \sigma &= -1 \quad , \end{aligned} \tag{37}$$

and the solution of the coupled space-time fractional mKdV equation is given by:

$$u(x,t) = k^{\alpha}\tanh_{\alpha}(kx+ct) \; , \; v(x,t) = \left(\lambda + 2k^{2\alpha}\right) - 2k^{2\alpha}\tanh_{\alpha}^2(kx+ct) \quad , \tag{38}$$

with:

$$c^{\alpha} = -k^{3\alpha} \quad . \tag{39}$$

For the second anzätz (Eq. (30)), setting $v_2 = 0$ into Eq. (33) and (34), the value of the parameters u_i , v_i (i=0,1), k and c, are given by:

$$\begin{aligned} u_1 &= -k^{\alpha} \quad , \\ u_0 &= -\frac{v_1}{2k^{\alpha}} \quad , \\ v_0 &= \lambda \quad , \\ c^{\alpha} &= k^{3\alpha}\sigma + 3k^{\alpha}u_0^2 \quad . \end{aligned} \tag{40}$$

We also know that the equation (27) have the kink-type soliton solution [41]:

$$u(x,t) = u_0 + u_1 \tanh_\alpha (kx + ct) \ , \quad v(x,t) = v_0 + b^\alpha \tanh_\alpha (kx + ct) \quad , \tag{41}$$

for the ansätz proposed in Eq. (30), where b is some constant parameter. We can compare this solution with Eq. (10) for the tanh-type solution and obtain the following values for the coefficients u_i and v_j, the parameters k, c and the constant σ:

$$\begin{aligned} u_1 &= -k^\alpha , \\ v_1 &= -2k^\alpha u_0 = 2u_1 u_0 , \\ v_1 &= -b^\alpha , \\ u_0 &= \frac{b^\alpha}{2k^\alpha}, \\ c^\alpha &= -k^{3\alpha} + \frac{3}{4} k^\alpha \left(\frac{b^\alpha}{k^\alpha} \right)^2 , \\ \sigma &= -1, \\ v_0 &= \lambda , \end{aligned} \tag{42}$$

and the solution to the coupled space-time fractional mKdV equation is given by:

$$u(x,t) = \frac{b^\alpha}{2k^\alpha} + k^\alpha \tanh_\alpha (kx + ct) \ , v(x,t) = \lambda + b^\alpha \tanh_\alpha (kx + ct) \quad , \tag{43}$$

with:

$$c^\alpha = -k^{3\alpha} + \frac{3}{4} k^\alpha \left(\frac{b^\alpha}{k^\alpha} \right)^2 \quad . \tag{44}$$

It should be noted that these results are in good agreement with previous results obtained by Zhao et al. [51], where they have applied the extended fractional sub-equation method to the mKdV fractional non-linear equation.

4. Discussion

The reliability of the sub-equation method to solve FPDEs has been demonstrated by applying this method to the space-time fractional Hirota-Satsuma coupled KdV equation. For the special case α=1 and σ=-1, we have successfully recovered the previously known solutions, equations (24) and (26), that have been found in Ref. [41].

However, we note that for the special case where α=1 the solutions (38) and (43) for the space-time fractional coupled mKdV equation show some differences with the solutions previously introduced by Fan [41]:

$$\tilde{u}(x,t) = k\tanh\left(kx + \tilde{c}t\right), \quad \tilde{v}(x,t) = \left(\frac{\lambda}{2} + 2k^2\right) - 2k^2\tanh^2\left(kx + \tilde{c}t\right) \quad , \tag{45}$$

$$\tilde{c} = \frac{k}{2}\left(-2k^2 - 3\lambda\right) \tag{46}$$

$$\tilde{u}(x,t) = \frac{b}{2k} + k\tanh\left(kx + \tilde{c}t\right), \tilde{v}(x,t) = \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + b\tanh\left(kx + \tilde{c}t\right) \quad , \tag{47}$$

$$\tilde{c} = -k^3 + \frac{3}{4}k\left(\frac{b}{k}\right)^2 + \frac{6}{4}k\lambda\left(\frac{k}{b} - 1\right) \quad . \tag{48}$$

The differences observed are important for the parameter c and the coefficient v_0, and we need to be sure that the solutions (38) and (43) here obtained are correct. For the case α=1 the solutions of the present work can be compared with the results obtained with the aid of some symbolic mathematical software, like *Mathematica*. The results obtained with *Mathematica* show that the solutions (38) and (43) are correct. The *Mathematica* code is given by:

```
In[1]:= u[x, t] := k Tanh[k x + c*t];
v[x, t] = (L+2.0*k^2) - 2.0*(k^2)*(Tanh[k x + c*t])^2;
In[2]:= FullSimplify[(1/2)*D[u[x, t], x, 3] - 3*(u[x, t])^2*D[u[x, t], x, 1]
+ (3/2)*D[v[x, t],x, 2] + 3*D[(u[x, t]*v[x, t]), x, 1]
- 3*L*D[u[x, t], x, 1]]
Out[2]= -1. k^4 Sech[c t + k x]^2
In[3]:= FullSimplify[D[u[x, t], t, 1]]
Out[3]= c k Sech[c t + k x]^2
In[4]:=FullSimplify[-D[v[x, t], x, 3] - 3*v[x, t]*D[(v[x, t]), x, 1]
-3*D[v[x, t], x, 1] * D[u[x, t], x, 1]
+ 3*(u[x, t])^2*D[v[x, t], x, 1] + 3*L*D[v[x, t], x, 1]]]
Out[4]= 4. k^5 Sech[c t + k x]^2 Tanh[c t + k x]
In[5]:=FullSimplify[D[v[x, t], t,1]];
Out[5]= -4. c k^2 Sech[c t + k x]^2 Tanh[c t + k x]
In[6]:=Solve[Out[4]/Out[5] == 1, c]
Out[6]={{c -> - k^3}}
In[7]:=u[x, t] := (b/(2*k)) + k Tanh[k x + c t];
v[x, t] = L + b*(Tanh[k x + c*t]);
In[8]:=FullSimplify[(1/2)*D[u[x, t], x, 3] - 3*(u[x, t])^2*D[u[x, t], x, 1]
+ (3/2)*D[v[x, t],x, 2] + 3*D[(u[x, t]*v[x, t]), x, 1]
- 3*L*D[u[x, t], x, 1]]
Out[8]= 1/4 (3 b^2 - 4 k^4) Sech[c t + k x]^2
In[9]:=FullSimplify[D[u[x, t], t, 1]]
Out[9]= c k Sech[c t + k x]^2
In[10]:=FullSimplify[-D[v[x, t], x, 3] - 3*v[x, t]*D[(v[x, t]), x, 1]
-3*D[v[x, t], x, 1] * D[u[x, t], x, 1]
+ 3*(u[x, t])^2*D[v[x, t], x, 1] + 3*L*D[v[x, t], x, 1]]]
```

```
Out[10]= (b (3 b^2 - 4 k^4) Sech[c t + k x]^2)/(4 k)
In[11]:=FullSimplify[D[v[x, t], t, 1]]
Out[11]= b c Sech[c t + k x]^2
In[12]:=Solve[Out[10]/Out[11] == 1, c]
Out[12]={{c -> (3 b^2 - 4 k^4)/(4 k)}}
```

5. Conclusion

In this paper we have investigated the exact travelling wave solutions to the space-time fractional Hirota–Satsuma KdV and the mKdV equations to illustrate the validity of the sub-equation method. For the KdV system, we have obtained the same solutions previously known. However, for the mKdV system, we found new exact solutions with important differences compared with the solutions obtained before [41]. We have confirmed the accuracy of the mKdV solutions here presented by using a symbolic code in *Mathematica*, for the special case when the fractional-order parameter for the mKdV coupled FPDE reaches its integer limit. These new exact solutions can be very useful as a starting point of comparison when some approximated methods are applied to the mKdV equation [46–49]. The present work demonstrates the wider applicability of the sub-equation method to nonlinear fractional coupled partial differential equations in mathematical physics.

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Competing Interests

The authors declare that no competing interests exist.

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