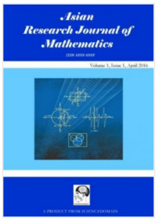

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On the Dual $\delta - k$–Fibonacci Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

We define two integer sequences that depend on a parameter δ and that are related to each other by two recurrence relations.

Then we find the Binet formula for the terms of these sequences and, by developing it, we will get an equivalent combinatorial formula.

We show that each sequence follows the same relation of recurrence although they differ in the initial conditions.

Later we show that these numbers are related to the k–Fibonacci numbers and we finish this section finding its generating functions.

Finally, for certain particular values of δ we show that these numbers are related to the Chebyshev polynomials.

This paper deals with a new concept of k–Fibonacci sequences linked to each other, so there is no literature on the subject. I hope that this article will be the starting point for other mathematicians that wish to investigate this topic.

Keywords: k–Fibonacci numbers; k–Lucas numbers; Binomial expansion; geometric sum; generating function; Chebyshev polynomials.

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1 Introduction

Classical Fibonacci numbers have been very used in as different Sciences as the Biology, Demography or Economy [1]. Recently they have been applied even in the high-energy physics [2, 3]. But there exist generalizations of these numbers given by researches as ([4]) who considers the generalized (a, b)– Fibonacci numbers as the numbers $F_n(a, b)$ verifying the recurrence relation $F_{n+1}(a, b) = a \cdot F_n(a, b) + b \cdot F_{n-1}(a, b)$ with the initial conditions $F_0(a, b) = p$ and $F_1(a, b) = q$.

In the study of the four-triangle longest-edge (4TLE) partition of a triangle [5], the k–Fibonacci numbers appear as a particular case of the generalized Fibonacci numbers. These numbers have been studied intensively in recent years [6].

Definition 1.1 (k–Fibonacci numbers)**.** For any positive real number k, the k–Fibonacci sequence, say $F_k = \{F_{k,n}\}_{n\in\mathbf{N}}$, is defined by the recurrence relation

$$F_{k,n+1} = k\ F_{k,n} + F_{k,n-1} \text{ for } n \geq 1$$

with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$

The sequence of the k–Fibonacci numbers is $F_k = \{0, 1, k, k^2 + 1, k^3 + 2k, \ldots\}$

For $k = 1$, classical Fibonacci sequence is obtained and for $k = 2$, Pell sequence appears.

We define the negative k–Fibonacci numbers as $F_{k,-n} = (-1)^{n+1}F_{k,n}$.

In similar form, the k–Lucas numbers are defined as $L_{k,n+1} = k\, L_{k,n} + L_{k,n-1}$ with initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$ [7].

The well–known Binet formula in the *Fibonacci numbers theory* [5, 4, 8] allows us to express the k–Fibonacci and the k–Lucas numbers by mean of the roots σ_1 and σ_2 of the characteristic equation associated to the recurrence relation $r^2 = k\, r + 1$. If $\sigma_{1,2} = \dfrac{k \pm \sqrt{k^2 + 4}}{2}$, $F_{k,n} = \dfrac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$ and $L_{k,n} = \sigma_1^n + \sigma_2^n$. As consequence $L_{k,n} = F_{k,n-1} + F_{k,n+1}$.

Shown below some properties of σ_1 and σ_2 that we will use in this paper:

$$\sigma_1 + \sigma_2 = k,\ \sigma_1 - \sigma_2 = \sqrt{k^2 + 4},\ \sigma_1 \cdot \sigma_2 = -1,\ \sigma^2 = k\,\sigma + 1,\ \sigma_1(\sigma_1 - k) = 1 \to k - \sigma_1 = -\frac{1}{\sigma_1} = \sigma_2.$$

2 Dual $\delta - k$–Fibonacci Sequences

In this section we define the dual $\delta - k$–Fibonacci sequences and study the relationships between them (for a general case, see [9, 10, 11]).
Let δ be a complex number.

Definition 2.1. We define the dual $\delta - k$–Fibonacci numbers $u_n(\delta)$ and $v_n(\delta)$, for $n \in \mathcal{N}$, in the form

$$u_n(\delta) - \sigma_1 v_n(\delta) = (1 - \sigma_1\delta)^n \tag{2.1}$$

or equivalently by the relation

$$u_n(\delta) - \sigma_2 v_n(\delta) = (1 - \sigma_2\delta)^n \tag{2.2}$$

While there is no confusion, we will represent $u_n(\delta)$ and $v_n(\delta)$ as u and v, respectively. Given values to n, we can find the first terms of the sequences $U(\delta) = \{u_n(\delta)\}$ and $V(\delta) = \{v_n(\delta)\}$. We must taking into account that $\sigma^n = F_{k,n}\sigma + F_{k,n-1}$. In particular, $\sigma^2 = k\,\sigma + 1$ and $\sigma^3 = (k^2+1)\sigma + k$.

$$\begin{array}{lll}
n=0 & \to & \left\{\begin{array}{l} u_0 - \sigma_1 v_0 = 1 \\ u_0 - \sigma_2 v_0 = 1 \end{array}\right\} \to (\sigma_1-\sigma_2)v_0 = 0 \to v_0 = 0 \to u_0 = 1 \\
n=1 & \to & \left\{\begin{array}{l} u_1 - \sigma_1 v_1 = 1 - \sigma_1\delta \\ u_1 - \sigma_2 v_1 = 1 - \sigma_2\delta \end{array}\right\} \to (\sigma_1-\sigma_2)v_1 = (\sigma_1-\sigma_2)\delta \\
 & & \qquad \to v_1 = \delta \to u_1 = 1 \\
n=2 & \to & \left\{\begin{array}{l} u_2 - \sigma_1 v_2 = 1 - 2\sigma_1\delta + \sigma_1^2\delta^2 \\ u_2 - \sigma_2 v_2 = 1 - 2\sigma_2\delta + \sigma_2^2\delta^2 \end{array}\right\} \\
 & & \qquad \to (\sigma_1-\sigma_2)v_2 = 2(\sigma_1-\sigma_2)\delta - (\sigma_1^2-\sigma_2^2)\delta^2 \\
 & & \qquad \to v_2 = 2\delta - k\,\delta^2 \to u_2 = 1 + \delta^2 \\
n=3 & \to & \cdots \to v_3 = 3\delta - 3k\,\delta^2 + (k^2+1)\delta^3 \to u_3 = 1 + 3\delta^2 - k\,\delta^3
\end{array}$$

Then,
$U(\delta) = \{1, 1, 1+\delta^2, 1+3\delta^2 - k\,\delta^3, 1+6d^2 - 4kd^3 + (1+k^2)d^4, \ldots\}$
$V(\delta) = \{0, \delta, 2\delta - k\,\delta^2, 3\delta - 3k\,\delta^2 + (k^2+1)\delta^3, 4d - 6kd^2 + 4(1+k^2)d^3 - (2k+k^3)d^4, \ldots\}$

We can see that the elements of these sequences verify the following relations that we will prove later (formulas (2.5) and (2.6)):
$u_{n+1} = u_n + \delta v_n, \qquad v_{n+1} = \delta u_n + (1 - k\,\delta)v_n$
Taking into account $u_0 = 1$ and $v_0 = 0$, by mean of these relations we can find easily the first elements of the sequences $U(\delta)$ and $V(\delta)$.

2.1 Binet formulas

Multiplying the equation (2.1) by $-\sigma_2$, the (2.2) by σ_1, and summing both results we obtain

$$u_n(\delta) = \frac{\sigma_1(1-\sigma_2\delta)^n - \sigma_2(1-\sigma_1\delta)^n}{\sigma_1 - \sigma_2} \tag{2.3}$$

Subtracting the equations (2.1) and (2.2), we obtain
$(\sigma_1 - \sigma_2)v_n = -(1-\sigma_1\delta)^n + (1-\sigma_2\delta)^n$ from where

$$v_n(\delta) = \frac{(1-\sigma_2\delta)^n - (1-\sigma_1\delta)^n}{\sigma_1 - \sigma_2} \tag{2.4}$$

2.2 Relations between $u_{n+1},\ v_{n+1}$ and $u_n,\ v_n$

Next we will prove the relations

$$u_{n+1} = u_n + \delta v_n \tag{2.5}$$
$$v_{n+1} = \delta u_n + (1 - k\,\delta)v_n \tag{2.6}$$

Proof of equation (2.5). From equation (2.3)

$$\begin{aligned}
u_{n+1} &= -\frac{\sigma_2(1-\sigma_1\delta)^{n+1}-\sigma_1(1-\sigma_2\delta)^{n+1}}{\sigma_1-\sigma_2}\\
&= -\frac{\sigma_2(1-\sigma_1\delta)(1-\sigma_1\delta)^{n}-\sigma_1(1-\sigma_2\delta)(1-\sigma_2\delta)^{n}}{\sigma_1-\sigma_2}\\
&= -\frac{(\sigma_2+\delta)(1-\sigma_1\delta)^{n}-(\sigma_1+\delta)(1-\sigma_2\delta)^{n}}{\sigma_1-\sigma_2}\\
&= -\frac{\sigma_2(1-\sigma_1\delta)^{n}-\sigma_1(1-\sigma_2\delta)^{n}}{\sigma_1-\sigma_2}-\delta\,\frac{(1-\sigma_1\delta)^{n}-(1-\sigma_2\delta)^{n}}{\sigma_1-\sigma_2}\\
&= u_n+\delta v_n
\end{aligned}$$

To prove the relation (2.6), we must take into account that $\sigma_1+\sigma_2=k$. Then

$$\begin{aligned}
&\delta u_n+(1-k\,\delta)v_n=\\
&=\frac{(\delta\sigma_1+1-k\,\delta)(1-\sigma_2\delta)^n-(\delta\sigma_2+1-k\,\delta)(1-\sigma_1\delta)^n}{\sigma_1-\sigma_2}\\
&=\frac{(1-(k-\sigma_1)\delta)(1-\sigma_2\delta)^n-(1-(k-\sigma_2)\delta)(1-\sigma_1\delta)^n}{\sigma_1-\sigma_2}\\
&=\frac{(1-\sigma_2\delta)(1-\sigma_2\delta)^n-(1-\sigma_1\delta)(1-\sigma_1\delta)^n}{\sigma_1-\sigma_2}\\
&=\frac{(1-\sigma_2\delta)^{n+1}-(1-\sigma_1\delta)^{n+1}}{\sigma_1-\sigma_2}=v_{n+1}
\end{aligned}$$

Then, with the initial conditions $u_0=1$ and $v_0=0$, it is relatively easy to find the previous sequences $U(\delta)$ and $V(\delta)$.

The terms of the sequence $U(\delta)$ (and $V(\delta)$) verify the following recurrence relation.

Theorem 2.1 (Recurrence relations in $U(\delta)$ and $V(\delta)$)**.**

$$u_{n+1}=(2-k\,\delta)u_n+(\delta^2+k\,\delta-1)u_{n-1} \tag{2.7}$$

$$v_{n+1}=(2-k\,\delta)v_n+(\delta^2+k\,\delta-1)v_{n-1} \tag{2.8}$$

To prove them, we will use equation (2.3) and must apply the following identity:

$$\begin{aligned}
&(2-k\,\delta)(1-\sigma_2\delta)+\delta^2+k\,\delta-1=\\
&=2-2\sigma_2\delta-k\,\delta+k\sigma_2\delta^2+\delta^2+k\,\delta-1=\\
&=1-2\sigma_2\delta+(k\sigma_2+1)\delta^2=1-2\sigma_2\delta^2+\sigma_2^2\delta^2=(1-\sigma_2\delta)^2
\end{aligned}$$

Proof of equation (2.7).

$$\begin{aligned}
u_n &= \frac{\sigma_1(1-\sigma_2\delta)^n-\sigma_2(1-\sigma_1\delta)^n}{\sigma_1-\sigma_2}\\
u_{n-1} &= \frac{\sigma_1(1-\sigma_2\delta)^{n-1}-\sigma_2(1-\sigma_1\delta)^{n-1}}{\sigma_1-\sigma_2}.\ \text{Then:}\\
&(2-k\,\delta)u_n+(\delta^2+k\,\delta-1)u_{n-1}=\\
&=\frac{(1-\sigma_2\delta)^{n-1}\sigma_1((2-k\,\delta)(1-\sigma_2\delta)+\delta^2+k\,\delta-1)}{\sigma_1-\sigma_2}\\
&-\frac{(1-\sigma_1\delta)^{n-1}\sigma_2((2-k\,\delta)(1-\sigma_1\delta)+\delta^2+k\,\delta-1)}{\sigma_1-\sigma_2}\\
&=\frac{(1-\sigma_2\delta)^{n+1}\sigma_1-(1-\sigma_1\delta)^{n+1}\sigma_2}{\sigma_1-\sigma_2}=u_{n+1}
\end{aligned}$$

In similar form we can prove the equation (2.8).

From these equations we can find the Binet formulas, taking into account that the recurrence equation of these formulas is $r^2-(2-k\,\delta)r-(\delta^2+k\,\delta-1)=0$. Finding the roots of this equation and remembering that $u_0=1,\ u_1=1$ and $v_0=0$, $v_1=\delta$, it is easy to find the formulas (2.3) and (2.4).

To find the u_n and v_n dual numbers it is easier to apply the following combinatorial formulas.

Theorem 2.2 (Relations between u_n, v_n, and the k–Fibonacci numbers)**.** *For* $n\geq 0$

$$\begin{aligned} u_n(\delta) &= \sum_{j=0}^{n}\binom{n}{j}F_{k,j-1}(-\delta)^j \qquad (2.9)\\ v_n(\delta) &= -\sum_{j=0}^{n}\binom{n}{j}F_{k,j}(-\delta)^j \end{aligned}$$

Proof.

We must taking into account that $\sigma_1^p\sigma_2=\sigma_1^{p-1}(\sigma_1\cdot\sigma_2)=-\sigma_1^{p-1}$. Then, expanding the Binet formula, it is

1.

$$\begin{aligned} u_n(\delta) &= \frac{1}{\sigma_1-\sigma_2}\left(-\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sigma_1^j\delta^j\sigma_2+\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sigma_2^j\delta^j\sigma_1\right)\\ &= \frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(\sigma_1^{j-1}-\sigma_2^{j-1}\right)\delta^j\\ u_n(\delta) &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}F_{k,j-1}\delta^j \end{aligned}$$

2.

$$\begin{aligned} v_n(\delta) &= \frac{1}{\sigma_1-\sigma_2}\left(-\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sigma_1^j\delta^j+\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sigma_2^j\delta^j\right)\\ &= \frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}(-1)^{j+1}\binom{n}{j}\left(\sigma_1^{j}-\sigma_2^{j}\right)\delta^j\\ v_n(\delta) &= \sum_{j=0}^{n}(-1)^{j+1}\binom{n}{j}F_{k,j}\delta^j \end{aligned}$$

That is: $\{v_(\delta)\}$ is the binomial transform of the sequence $\{-F_{k,jn}(-1)^n\}$.

By using of Mathematica® and applying the preceding formulas, we can obtain the six first terms of the sequences $\{u_n(\delta)\}$ and $\{v_n(\delta)\}$ in the following form:

f[k_ , n_] : = Fibonacci[n, k]

$u[k\_ ,\ n\_ ,\ \delta\_]\ :=\ \sum_{j=0}^{n}$ Binomial$[n,\ j]\ f[k,\ j-1]\ (-\delta)^j$

$v[k\_ ,\ n\_ ,\ \delta\_]\ :=\ \sum_{j=0}^{n}$ Binomial$[n,\ j]\ f[k,\ j]\ (-\delta)^j$

Table[Expand[$u[k\ ,\ n\ ,\ \delta]$], $\{n,0,5\}$]

Table[Expand[$v[k\ ,\ n\ ,\ \delta]$], $\{n,0,5\}$]

The δ symbol is written in Mathematica as \[*Delta*]

2.3 New combinatorial formulas for u_n and v_n

Later, and using the Binet identities, we will prove the following two new formulas for u_n and v_n, respectively:

$$u_n(\delta) = \sum_{j=0}^{n} \binom{n}{j} F_{k,j+1}(1 - k\,\delta)^{n-j}\delta^j \tag{2.10}$$

$$v_n(\delta) = \sum_{j=0}^{n} \binom{n}{j} F_{k,j}(1 - k\,\delta)^{n-j}\delta^j \tag{2.11}$$

To prove the equation (2.10), we must remember that $\sigma_1 + \sigma_2 = k$

$$\begin{aligned}
&\sum_{j=0}^{n} \binom{n}{j} F_{k,j+1}(1 - k\,\delta)^{n-j}\delta^j \\
&= (1 - k\,\delta)^n \sum_{j=0}^{n} \binom{n}{j} \frac{\sigma_1^{j+1} - \sigma_2^{j+1}}{\sigma_1 - \sigma_2}(1 - k\,\delta)^{-j}\delta^j \\
&= \frac{(1 - k\,\delta)^n}{\sigma_1 - \sigma_2}\left[\sigma_1 \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\sigma_1\delta}{1 - k\,\delta}\right)^j - \sigma_2 \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\sigma_2\delta}{1 - k\,\delta}\right)^j\right] \\
&= \frac{(1 - k\,\delta)^n}{\sigma_1 - \sigma_2}\left[\sigma_1 \left(1 + \frac{\sigma_1\delta}{1 - k\,\delta}\right)^n - \sigma_2 \left(1 + \frac{\sigma_2\delta}{1 - k\,\delta}\right)^n\right] \\
&= \frac{(1 - k\,\delta)^n}{\sigma_1 - \sigma_2}\left[\sigma_1 \frac{(1 + (\sigma_1 - k)\delta)^n}{(1 - k\,\delta)^n} - \sigma_2 \frac{(1 + (\sigma_2 - k)\delta)^n}{(1 - k\,\delta)^n}\right] \\
&= \frac{\sigma_1(1 - \sigma_2\delta)^n - \sigma_2(1 - \sigma_1\delta)^n}{\sigma_1 - \sigma_2} = u_n(\delta)
\end{aligned}$$

In the same way we prove the formula (2.11).

Theorem 2.3 (Generating function of the sequence $U(\delta)$)**.** *The generating function of the sequence $U(\delta)$ is*

$$u(x) = \frac{1 + (k\,\delta - 1)x}{1 - (2 - k\,\delta)x - (\delta^2 + k\,\delta - 1)x^2}$$

Proof.
Taking into account the recurrence relations (2.7) and (2.8),

$$\begin{aligned}
u(x) &= u_0 + u_1(x) + u_2x^2 + u_3x^3 + \cdots \\
(2 - k\,\delta)x\,u(x) &= (2 - k\,\delta)u_0x + (2 - k\,\delta)u_1x^2 + (2 - k\,\delta)u_2x^3 + \cdots \\
(\delta^2 + k\,\delta - 1)x^2u(x) &= (\delta^2 + k\,\delta - 1)u_0x^2 + (\delta^2 + k\,\delta - 1)u_1x^3 + \cdots
\end{aligned}$$

from where, the first equation, less the second and third equations, gives
$\left(1 - (2 - k\,\delta)x - (\delta^2 + k\,\delta - 1)x^2\right) u(x) = u_0 + (u_1 - (2 - k\,\delta)u_0)\,x \rightarrow$
$\rightarrow u(x) = \dfrac{1 + (k\,\delta - 1)x}{1 - (2 - k\,\delta)x - (\delta^2 + k\,\delta - 1)x^2}$

The generating function of the sequence $V(\delta)$ can be found in the same way, but we prefer to do it in a different way.

Theorem 2.4 (Generating function of the sequence $V(\delta)$)**.** *The generating function of the sequence $V(\delta)$ is*

$$v(x) = \frac{\delta\,x}{1 - (2 - k\delta\,)x - (\delta^2 + k\delta - 1)x^2}$$

Proof.
In [12] is proven the following theorem: Let s, t be given complex numbers and let $\{A_n\}_{n=0}^{\infty}$ be a given sequence of numbers and $A(x)$ its generating function.
If $z(n) = \sum_{j=0}^{n} \binom{n}{j} t^{n-j} s^j A_j$, the generating function of the sequence $\{z_n\}$ is $Z(x) = \frac{1}{1-t\,x} A\left(\frac{s\,x}{1-t\,x}\right)$.
We apply this theorem to the formula (2.11), $v_n = \sum_{j=0}^{n} \binom{n}{j} F_{k,j}(1-k\,\delta)^{n-j}\delta^j$ with $A_j = F_{k,j}$, $t = 1 - k\,\delta$, and $s = \delta$, and being $f(x = \frac{x}{1-k\,x-x^2})$ the generating function of the sequence $F_k = \{F_{k,n}\}$. Then, the generating function of the sequence V is

$$
\begin{aligned}
v(x) &= \frac{1}{1-(1-k\,\delta)x} f\left(\frac{\delta x}{1-(1-k\,\delta)x}\right) \\
&= \frac{1}{1-(1-k\,\delta)x} \cdot \frac{\frac{\delta\,x}{1-(1-k\,\delta)x}}{1-k\left(\frac{\delta\,x}{1-(1-k\,\delta)x}\right)-\left(\frac{\delta\,x}{1-(1-k\,\delta)x}\right)^2} \\
&= \frac{\delta\,x}{1-2x+k\delta\,x+x^2-\delta^2x^2-k\delta\,x^2} \to \\
v(x) &= \frac{\delta\,x}{1-(2-k\delta\,)x-(\delta^2+k\delta-1)x^2}
\end{aligned}
$$

Since the $\delta - k$–Fibonacci numbers are in fact the binomial transformations of the scaled sequences of Fibonacci, thus in some moment we realized that it would be welcome, for emphasizing the meaning of these numbers, to have at our disposal the general formulas for $\delta - k$–Fibonacci numbers for "the most generally expressed" parameters δ from the set of complex numbers. So, these are the roots of this work [13].

By using of Mathematica® we can get the first six terms of the sequences $\{u_n(\delta)\}$ and $\{v_n(\delta)\}$ again in the following way:

$u[x\_] := \frac{1 + (k\ \delta - 1)\ x}{1 - (2 - k\ \delta)\ x - (\delta^2 + k\ \delta - 1)\ x^2}$

$v[x\_] := \frac{\delta\ x}{1 - (2 - k\ \delta)\ x - (\delta^2 + k\ \delta - 1)\ x^2}$

CoefficientList[Series[u[x], {x, 0, 5}], x]
CoefficientList[Series[v[x], {x, 0, 5}], x]

3 Main Theorems and Results

In this section we will study the properties of the $\delta - k$–Fibonacci numbers for particular cases of δ.

Theorem 3.1.

$$\textit{If } \delta = \frac{F_{k,p}}{F_{k,p+1}} : u_n = \frac{F_{k,p\,n+1}}{F_{k,p+1}^n}, \quad v_n = \frac{F_{k,p\,n}}{F_{k,p+1}^n}$$

Proof. If $\delta = \frac{F_{k,p}}{F_{k,p+1}} \to 1 - k\,\delta = \frac{F_{k,p-1}}{F_{k,p+1}}$. From the equation (2.10) and taking into account that $F_{k,p}\sigma + F_{k,p-1} = \sigma^p$,

$$\begin{aligned}
u_n &= \sum_{j=0}^{n} \binom{n}{j} \frac{\sigma_1^{j+1} - \sigma_2^{j+1}}{\sigma_1 - \sigma_2} \left(\frac{F_{k,p-1}}{F_{k,p+1}}\right)^{n-j} \left(\frac{F_{k,p}}{F_{k,p+1}}\right)^{j} \\
&= \left(\frac{F_{k,p-1}}{F_{k,p+1}}\right)^{n} \frac{1}{\sigma_1 - \sigma_2} \left(\sigma_1 \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\sigma_1 F_{k,p}}{F_{k,p-1}}\right)^{j} - \sigma_2 \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\sigma_2 F_{k,p}}{F_{k,p-1}}\right)^{j}\right) \\
&= \left(\frac{F_{k,p-1}}{F_{k,p+1}}\right)^{n} \frac{1}{\sigma_1 - \sigma_2} \left(\sigma_1 \left(1 + \frac{\sigma_1 F_{k,p}}{F_{k,p-1}}\right)^{n} - \sigma_2 \left(1 + \frac{\sigma_2 F_{k,p}}{F_{k,p-1}}\right)^{n}\right) \\
&= \frac{1}{F_{k,p+1}^n} \frac{\sigma_1 (F_{k,p-1} + \sigma_1 F_{k,p})^n - \sigma_2 (F_{k,p-1} + \sigma_2 F_{k,p})^n}{\sigma_1 - \sigma_2} \\
&= \frac{1}{F_{k,p+1}^n} \left(\frac{\sigma_1^{p\,n+1} - \sigma_2^{p\,n+1}}{\sigma_1 - \sigma_2}\right) = \frac{1}{F_{k,p+1}^n} F_{k,p\,n+1}
\end{aligned}$$

We prove the second formula in the same way.
In particular, if $p = 2 \to \delta = \frac{k}{k^2+1}$. Then $u_n\left(\dfrac{k}{k^2+1}\right) = \dfrac{F_{k,2n+1}}{(k^2+1)^n}$ and $v_n\left(\dfrac{k}{k^2+1}\right) = \dfrac{F_{k,2n}}{(k^2+1)^n}$

Theorem 3.2 ($e^{i\alpha}$ and the $\delta - k$–Fibonacci and Lucas numbers)**.** *For* $\delta = -\dfrac{1}{2}\left(\sigma_1 + \sigma_2 + i(\sigma_1 - \sigma_2)\tan\alpha\right)$ *and* $\alpha \notin \{(2n+1)\frac{\pi}{2}\}$,

$$v_n(\delta) = \left(\frac{\sigma_1 - \sigma_2}{2\cos\alpha}\right)^n \frac{(-1)^n \sigma_2^n e^{-in\alpha} - \sigma_1^n e^{in\alpha}}{\sigma_1 - \sigma_2} \tag{3.1}$$

$$u_n(\delta) = \left(\frac{\sigma_1 - \sigma_2}{2\cos\alpha}\right)^n \frac{(-1)^{n+1} \sigma_2^{n-1} e^{-in\alpha} + \sigma_1^{n-1} e^{in\alpha}}{\sigma_1 - \sigma_2} \tag{3.2}$$

Proof. We are going to find the value of the right member of equation (2.4).

$$\begin{aligned}
\delta &= -\frac{1}{2\cos\alpha}\left((\sigma_1 + \sigma_2)\cos\alpha + i(\sigma_1 - \sigma_2)\sin\alpha\right) \\
1 - \sigma_2\delta &= 1 + \frac{\sigma_2}{2\cos\alpha}\left((\sigma_1 + \sigma_2)\cos\alpha + i(\sigma_1 - \sigma_2)\sin\alpha\right) \\
&= \frac{1}{2\cos\alpha}\left(2\cos\alpha + (-1 + \sigma_2^2)\cos\alpha + i(-1 - \sigma_2^2)\sin\alpha\right) \\
&= \frac{1}{2\cos\alpha}\left(2\cos\alpha + k\,\sigma_2\cos\alpha - i(k\,\sigma_2 + 2)\sin\alpha\right) \\
&= \frac{1}{2\cos\alpha}\left(2(\cos\alpha - i\sin\alpha) + k\,\sigma_2(\cos\alpha - i\sin\alpha)\right) \\
&= \frac{1}{2\cos\alpha}(2 + k\,\sigma_2)e^{-i\alpha} \\
&\to\ 2 + k\,\sigma_2 = -(2\sigma_1 - k)\sigma_2 = -(2\sigma_1 - \sigma_1 - \sigma_2)\sigma_2 = -(\sigma_1 - \sigma_2)\sigma_2 \\
&\quad\ 1 - \sigma_2\delta = -\frac{\sigma_1 - \sigma_2}{2\cos\alpha}\sigma_2 e^{-i\alpha} \\
&\to\ (1 - \sigma_2\delta)^n = (-1)^n\left(\frac{\sigma_1 - \sigma_2}{2\cos\alpha}\right)^n \sigma_2^n e^{-in\alpha} \\
&\quad\ (1 - \sigma_1\delta)^n = \left(\frac{\sigma_1 - \sigma_2}{2\cos\alpha}\right)^n \sigma_1^n e^{in\alpha} \\
&\quad\ (1 - \sigma_2\delta)^n - (1 - \sigma_1\delta)^n = \left(\frac{\sigma_1 - \sigma_2}{2\cos\alpha}\right)^n \left((-1)^n \sigma_2^n e^{-in\alpha} - \sigma_1^n e^{in\alpha}\right) \\
&\to\ v_n(\delta) = \left(\frac{\sigma_1 - \sigma_2}{2\cos\alpha}\right)^n \frac{(-1)^n \sigma_2^n e^{-in\alpha} - \sigma_1^n e^{in\alpha}}{\sigma_1 - \sigma_2}
\end{aligned}$$

In the same way we can prove the formula (3.2) from the formula (2.3).

We are going to develop the equations (3.1) and (3.2) for the case where n is even or odd.

1. For even $n \in \mathcal{N}$:

$$\begin{aligned} v_n(\delta) &= \left(\frac{\sigma_1-\sigma_2}{2\cos\alpha}\right)^n \frac{(-1)^n\sigma_2^n e^{-in\alpha}-\sigma_1^n e^{in\alpha}}{\sigma_1-\sigma_2} \\ &= \left(\frac{\sigma_1-\sigma_2}{2\cos\alpha}\right)^n \frac{\sigma_2^n(\cos(n\alpha)-i\sin(n\alpha))-\sigma_1^n(\cos(n\alpha)+i\sin(n\alpha))}{\sigma_1-\sigma_2} \\ &= -\left(\frac{\sigma_1-\sigma_2}{2\cos\alpha}\right)^n \left(\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}\cos(n\alpha)+i\frac{\sigma_1^n+\sigma_2^n}{\sigma_1-\sigma_2}\sin(n\alpha)\right) \\ &= -\left(\frac{\sqrt{k^2+4}}{2\cos\alpha}\right)^n \left(F_{k,n}\cos(n\alpha)+i\frac{L_{k,n}}{\sqrt{k^2+4}}\sin(n\alpha)\right) \end{aligned}$$

2. For odd $n \in \mathcal{N}$:

$$\begin{aligned} v_n(\delta) &= -\left(\frac{\sigma_1-\sigma_2}{2\cos\alpha}\right)^n \frac{\sigma_2^n e^{-in\alpha}+\sigma_1 e^{in\alpha}}{\sigma_1-\sigma_2} \\ &= -\left(\frac{\sigma_1-\sigma_2}{2\cos\alpha}\right)^n \frac{\sigma_2^n(\cos(n\alpha)-i\sin(n\alpha))+\sigma_1^n(\cos(n\alpha)+i\sin(n\alpha))}{\sigma_1-\sigma_2} \\ &= -\left(\frac{\sqrt{k^2+4}}{2\cos\alpha}\right)^n \left(\frac{L_{k,n}}{\sqrt{k^2+4}}\cos(n\alpha)+iF_{k,n}\sin(n\alpha)\right) \end{aligned}$$

Similarly

1. For even $n \in \mathcal{N}$:

$$u_n(\delta) = \left(\frac{\sqrt{k^2+4}}{2\cos\alpha}\right)^n \left(F_{k,n-1}\cos(n\alpha)+i\frac{L_{k,n-1}}{\sqrt{k^2+4}}\sin(n\alpha)\right)$$

2. For odd $n \in \mathcal{N}$:

$$u_n(\delta) = \left(\frac{\sqrt{k^2+4}}{2\cos\alpha}\right)^n \left(\frac{L_{k,n-1}}{\sqrt{k^2+4}}\cos(n\alpha)+iF_{k,n-1}\sin(n\alpha)\right)$$

3.1 Relation between the $\delta - k$–Fibonacci and Lucas numbers and the Chebyshev polynomials

Let $(\sigma_1-\sigma_2)\tan\alpha = (\sigma_1+\sigma_2)\tan\beta$ be. Evidently $\alpha = \arctan\left(\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}\tan\beta\right)$. Then

$$\begin{aligned} \delta &= -\frac{k+ik\tan\beta}{2} = -\frac{k}{2\cos\beta}(\cos\beta+i\sin\beta) = -\frac{k}{2\cos\beta}e^{i\beta} \\ \cos\alpha &= \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+\left(\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}\tan\beta\right)^2}} \\ &= \frac{\sqrt{k^2+4}\cos\beta}{\sqrt{(k^2+4)\cos^2\beta+k^2\sin^2\beta}} = \frac{\sqrt{k^2+4}\cos\beta}{\sqrt{k^2+4\cos^2\beta}} \\ \sin\alpha &= \tan\alpha\cdot\cos\alpha = \frac{k}{\sqrt{k^2+4}}\frac{\sin\beta}{\cos\beta}\cdot\frac{\sqrt{k^2+4}\cos\beta}{\sqrt{k^2+4\cos^2\beta}} = \frac{k\sin\beta}{\sqrt{k^2+4\cos^2\beta}} \end{aligned}$$

1. $\beta = \frac{\pi}{3} \to \cos\alpha = \frac{1}{2}\frac{\sqrt{k^2+4}}{\sqrt{k^2+1}}$, $\sin\alpha = \frac{1}{2}\frac{\sqrt{3k^2}}{\sqrt{k^2+1}}$, $\delta = -ke^{i\frac{\pi}{3}}$, from where
$v_n\left(-k\,e^{i\,\pi/3}\right) = -\left(\sqrt{k^2+4}\right)^n\left(F_{k,n}T_n(\cos\alpha) + \frac{i}{\sqrt{k^2+4}}L_{k,n}U_{n-1}(\cos\alpha)\right)$
being $T_n(\cos\alpha)$ and $U_n(\cos\alpha$ the Chebyshev polynomials of first and second kind, respectively ([14]).
Furthermore, in [15, 16] are proven two binomial formulas for $T_n(x)$ and $U_n(x)$, respectively:

$$
\begin{aligned}
&T_n(x) = \frac{n}{2}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{(-1)^j}{n-j}\binom{n-j}{j}(2x)^{n-2j} \to \\
\to\ &T_{2n}\left(\frac{1}{2}\sqrt{\frac{k^2+4}{k^2+1}}\right) = n\sum_{j=0}^{n}\frac{(-1)^j}{2n-j}\binom{2n-j}{j}\left(\frac{k^2+4}{k^2+1}\right)^{n-j} \\
&T_{2n-1}\left(\frac{1}{2}\sqrt{\frac{k^2+4}{k^2+1}}\right) = \\
&= \frac{2n-1}{2}\sqrt{\frac{k^2+1}{k^2+4}}\sum_{j=0}^{n-1}\frac{(-1)^j}{2n-1-j}\binom{2n-1-j}{j}\left(\frac{k^2+4}{k^2+1}\right)^{n-j}
\end{aligned}
$$

Similar formulas hold for the Chebyshev polynomials of the second kind:

$$
\begin{aligned}
&U_n(x) = \frac{n}{2}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}(2x)^{n-2j} \to \\
\to\ &U_{2n}\left(\frac{1}{2}\sqrt{\frac{k^2+4}{k^2+1}}\right) = n\cdot\sum_{j=0}^{n}(-1)^j\binom{2n-j}{j}\left(\frac{k^2+4}{k^2+1}\right)^{n-j} \\
&U_{2n-1}\left(\frac{1}{2}\sqrt{\frac{k^2+4}{k^2+1}}\right) = \\
&= \frac{2n-1}{2}\cdot\sqrt{\frac{k^2+1}{k^2+4}}\sum_{j=0}^{n-1}(-1)^j\binom{2n-1-j}{j}\left(\frac{k^2+4}{k^2+1}\right)^{n-j}
\end{aligned}
$$

2. $\beta = \frac{\pi}{4} \to \cos\alpha = \frac{1}{\sqrt{2}}\frac{\sqrt{k^2+2}}{\sqrt{k^2+1}}$, $\sin\alpha = \frac{k}{\sqrt{2}\sqrt{k^2+2}}$, $\alpha = \arctan\left(\frac{k}{\sqrt{k^2+4}}\right)$ and then $v_n\left(-\frac{k}{\sqrt{2}}e^{i\,\pi/4}\right) = -\left(\frac{k^2+4}{2}\right)^{\frac{n}{2}}\left(F_{k,n}T_n(\cos\alpha) + i\frac{L_{k,n}}{\sqrt{k^2+4}}U_{n-1}(\cos\alpha)\right)$. For $x = \frac{1}{\sqrt{2}}\sqrt{\frac{k^2+2}{k^2+1}}$ it is easy to find the Chebyshev polynomials $T_{2n}(x)$, $T_{2n-1}(x)$, $U_{2n}(x)$, and $U_{2n-1}(x)$.

3. $\beta = \frac{\pi}{6} \to \cos\alpha = \frac{\sqrt{3}}{2}\frac{\sqrt{k^2+4}}{\sqrt{k^2+1}}$, $\sin\alpha = \frac{1}{\sqrt{2}}\sqrt{\frac{k^2-4}{k^2+2}}$, $\alpha = \arctan\left(\frac{\sqrt{3(k^2+4)}}{3k}\right)$ and then $v_n\left(-\frac{k}{\sqrt{3}}e^{i\,\pi/6}\right) = -\left(\frac{k^2+4}{3}\right)^{\frac{n}{2}}\left(F_{k,n}T_n(\cos\alpha) + i\frac{L_{k,n}}{\sqrt{k^2+4}}U_{n-1}(\cos\alpha)\right)$, with $\cos\alpha = \sqrt{\frac{3(k^2+4)}{4(k^2+1)}}$, and we can find the respective Chebyshev polynomials of first and second kind for $2n$ and $2n-1$, respectively.

3.2 If $\delta = -i$

To prove the following identities, we must taking into account this equality: $(1+i\sigma)^2 = 1+2i\sigma-\sigma^2 = 1+2\sigma i-(k\sigma+1) = (2i-k)\sigma$

- From equation (2.7): $u_{n+1}(-i) = (2+i\,k)(u_n(-i) - u_{n-1}(-i))$
- $u_{2n+2}(-i) = (2i-k)^{n+1}F_{k,n}$.
  Proof. From equation (2.3).
  $\sigma_1(1+i\sigma_2)^{2n+2} = \sigma_1(1+i\sigma_2)^2((1+i\sigma_2)^2)^n = \sigma_1(2i-k)\sigma_2(2i-k)^n\sigma_2^n = -(2i-k)^{n+1}\sigma_2^n$ and $\sigma_2(1+i\sigma_1)^{2n+2} = -(2i-k)^{n+1}\sigma_1^n$.
  Then, $u_{2n+2}(-i) = \dfrac{\sigma_1(1+i\sigma_2)^n - \sigma_2(1+i\sigma_1)^n}{\sigma_1-\sigma_2} = (2i-k)^{n+1}\dfrac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}$
  $= (2i-k)^{n+1}F_{k,n}$
- $U_{2n+3}(-i) = (2i-k)^{n+1}\left(F_{k,n} + i\,F_{k,n+1}\right)$.
  Proof. From equation (2.9),

$$u_{2n+3}(-i) = \sum_{j=0}^{2n+3}\binom{2n+3}{j}F_{k,j-1}i^j$$

$$\begin{aligned}
&= \frac{1}{\sigma_1-\sigma_2}\left(\frac{1}{\sigma_1}\sum_{j=0}^{2n+3}\binom{2n+3}{j}(i\sigma_1)^j - \frac{1}{\sigma_2}\sum_{j=0}^{2n+3}\binom{2n+3}{j}(i\sigma_2)^j\right)\\
&= \frac{1}{\sigma_1-\sigma_2}\left(\frac{1}{\sigma_1}(1+i\,\sigma_1)^{2n+3} - \frac{1}{\sigma_2}(1+i\,\sigma_2)^{2n+3}\right)\\
&= \frac{1}{\sigma_1-\sigma_2}\left(\frac{1}{\sigma_1}(1+i\,\sigma_1)(1+i\,\sigma_1)^2((1+i\,\sigma_1)^2)^n\right.\\
&\qquad \left. -\frac{1}{\sigma_2}(1+i\,\sigma_2)(1+i\,\sigma_2)^2((1+i\,\sigma_2)^2)^n\right)\\
&= \frac{1}{\sigma_1-\sigma_2}\left((1+i\,\sigma_1)(2i-k)((2i-k)\sigma_1)^n\right.\\
&\qquad \left. -(1+i\,\sigma_2)(2i-k)((2i-k)\sigma_2)^n\right)\\
&= (2i-k)^{n+1}\left(\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2} + i\,\frac{\sigma_1^{n+1}-\sigma_2^{n+1}}{\sigma_1-\sigma_2}\right)\\
&= (2i-k)^{n+1}\left(F_{k,n} + i\,F_{k,n+1}\right)
\end{aligned}$$

Similarly,
$v_{n+1}(-i) = (2+i\,k)\,(v_n(-i) - v_{n-1}(-i))$
$v_{2n}(-i) = (2i-k)^n F_{k,n}$
$v_{2n+1}(-i) = -(2i-k)^n\left(F_{k,n} + i\,F_{k,n+1}\right)$

4 Conclusions

We have defined two dual δ sequences by mean of two relations between them (2.1) and (2.2) and then we prove that they are two generalized Fibonacci sequences. We find later the Binet formulas of these numbers, and by developing these equations, we see that these numbers $u_n(\delta)$ and $v_n(\delta)$ are the binomial transforms of $F_{k,n-1}(-\delta)^n$ and $F_{k,n}(-\delta)^n$, respectively (equations 2.7 and 2.8). For a particular value of the parameter δ, we are able to relate them to $e^{i\alpha}$ (equations 3.1, 3.2). And finally, by mean of the characteristic roots $\sigma_{1,2}$, we see that some of these last numbers are related to the Chebyshev polynomials.

The way remains open to continue investigating these sequences and their properties.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Hogat VE. Fibonacci and Lucas numbers. Palo Alto: Houghton-Mifflin; 1969.

[2] El Naschie MS. Notes on superstrings and the infinite sums of Fibonacci and Lucas numbers. Chaos, Solit. & Fract. 2001;12(10):1937-1940.

[3] El Naschie MS. E-eight exceptional Lie groups, Fibonacci lattices and the standard model. Chaos, Solit. & Fract; 2009. doi:10.1016/j.chaos.2008.05.015.

[4] Horadam AF. A generalized Fibonacci sequence. Math. Mag. 1961;68:455-459.

[5] Falcon S, Plaza A. On the Fibonacci k–numbers. Chaos, Solitons & Fractals; 2006. DOI:10.1016/j.chaos.2006.09.022.

[6] Falcon S, Plaza A. The k-Fibonacci sequence and the Pascal 2-triangle. Chaos, Solitons & Fractals. 2007;33(1):38-49.

[7] Falcon S. On the k-Lucas numbers. International Journal of Contemporary Mathematical Sciences. 2011;6(21):1039-1050.

[8] Spinadel VW. The family of metallic means. Vis Math. 1999;1(3). Available:http://members.tripod.com/vismath/.

[9] Witula R. i-Fibonacci numbers. Part II, Novi Sad Math. J. 43(1):9-22.

[10] Witula R, Slota D. δ–Fibonacci numbers. Appl. Anal. Discrete Math. 2009;3:310-329.

[11] Witula R, Hetmaniok E, Slota D, Pleszczynski M. δ–Fibonacci and δ–Lucas numbers, δ–Fibonacci and δ–Lucas polynomials. Mathematica Slovaca. 67(1):51-70.

[12] Haukkanen P. Formal power series for binomial sums of sequences of numbers. The Fibonacci Quarterly. 1993;31(1):28-31.

[13] Hetmaniok E, Piatek B, Witula R. Binomial transformation formulae for scaled Fibonacci numbers. Open Mathematics. 2017;15(1):477-485. DOI:https://doi.org/10.1515/math-2017-0047

[14] Available:https://en.wikipedia.org/wiki/Chebyshev_polynomials

[15] Paszkowski S. Numerical applications of Chebyshev polynomials and series. Warsaw: PWN, (in Polish); 1975.

[16] Rivlin T. Chebyshev polynomials from approximation theory to algebra and number theory. New York: Willey; 1990.

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