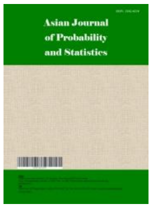

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Estimating Stress-Strength Model for Weighted Lomax Distribution

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Authors' contributions

This work was carried out in collaboration among all authors. Author MMEAEM designed the study, performed the statistical analysis and wrote the first draft of the manuscript. Author GAM managed the analyses of the study. Author NMK wrote the protocol and managed the literature searches. All authors read and approved the final manuscript.

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Abstract

This paper aims to estimate the stress-strength reliability parameter $R = P(Y < X)$, when X and Y are follow the weighted Lomax (WL) distribution. The behavior of stress-strength parameters and reliability have been studied by using maximum likelihood and Bayesian estimators through the Monte Carlo simulation study which carried out showing satisfactory performance of the estimators obtained. Finally, two real data sets representing waiting times before service of the customers of two banks A and B are fitted using the WL distribution and used to estimate the stress-strength parameters and reliability function.

Keywords: Weighted Lomax distribution; stress-strength reliability; maximum likelihood estimates; Bayes estimates.

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1 Introduction

The estimation of stress-strength parameter plays an important role in the reliability analysis. For example, if X is the strength of a system which is subjected to stress Y, then the parameter R measures the system performance which is frequently used in the context of mechanical reliability of a system. Moreover, R provides the probability of a system failure, if the system fails whenever the applied stress is greater than its strength. The stress-strength reliability R is then defined as the probability of not failing i.e. $P[X > Y]$.

The idea of stress strength reliability has been originally proposed by Birnbaum [1]. For instance, the estimation of R when X and Y are independent and normally-distributed has been considered by several authors including Mokhlis [2], Milan and Vesna [3] and Greco and Venture [4] reported a list of papers related to the estimation problem of R when X and Y are independent and follow a class of lifetime distributions including exponential, bivariate exponential, generalized exponential, Gamma distributions, Burr type I model, Weibull distribution, and other.

Nowadays the stress-strength model is of substantial interest and usefulness in various areas of science. Different examples of applications of $P[X > Y]$ in engineering and medicine is presented in [5] and the monograph by Kotz, [6].

The main aim of this paper is estimating procedure for the stress-strength parameter $R = P(Y < X)$, when X and Y are independent $WL(\alpha_1, \lambda_1, \beta_1)$ and $WL(\alpha_2, \lambda_2, \beta_2)$ respectively.

The WL distribution with three parameters (α, λ, β) which introduced by Kilany [7] with probability density function as follows

$$f(x) = \frac{\Gamma(\alpha+1)\lambda^{1+\alpha-\beta}}{\Gamma(1+\alpha-\beta)\Gamma(\beta)}\left(\frac{x^{\beta-1}}{(x+\lambda)^{\alpha+1}}\right),\ x \geq 0\,, \lambda > 0, \alpha > 0, 0 < \beta < \alpha + 1, \tag{1}$$

where Γ(·) is the compelet gamma function. Note that, when $\beta = 1$, the WL distribution reduces to the Lomax distribution. The corresponding cumulative distribution function can be written as

$$F(x) = \frac{\Gamma(\alpha+1)\lambda^{-\beta}x^{\beta} \times {}_2F_1(\alpha+1,\beta,\beta+1;-\frac{x}{\lambda})}{\beta\Gamma(\beta)\Gamma(1+\alpha-\beta)}\ , \tag{2}$$

where ${}_2F_1(a,b,c;z)$ is the hypergeometric function.

The reliability parameter for WL distribution is given by

$$R = P(Y < X) = \int_0^\infty \int_0^x f(y;\alpha_2,\lambda_2,\beta_2)\, f(x;\alpha_1,\lambda_1,\beta_1)\, dydx$$

$\mathrm{R} = \Big\{\Big[\frac{(\lambda_2/\lambda_1)^{\alpha_2-\beta_2+1}\Gamma(\alpha_2+1)\Gamma(2+\alpha_1+\alpha_2-\beta_1-\beta_2)\Gamma(-1-\alpha_2+\beta_1+\beta_2)}{(\alpha_2-\beta_2+1)\Gamma(\beta_1)\Gamma(\beta_2)(\alpha_1-\beta_1+1)\Gamma(\alpha_2-\beta_2+1)} \times\ {}_p\mathrm{F}_q(\{1+\alpha_2, 1+\alpha_2-\beta_2, 2+\alpha_1+\alpha_2-\beta_1-$
β2},{2+α2−β2,2+α2−β1−β2};λ2λ1)+(λ2/λ1)β1Γα1+1Γ1+α2−β1−β2Γβ1+β2(α2−β2+1)Γβ1Γβ2α1−β1+1Γα2−β2+1× pFq(1+α1,β1,β1+β2,1+β1,−α2+β1+β2;λ2λ1),
(3)

where ${}_pF_q(\{a_1, a_2, \dots, a_p\}, \{b_1, b_2, \dots, b_1\}; z)$ is the generalized hypergeometric function.

The paper has been organized in the following sections. In Section 2, the estimation of R are studied for the general case when the parameters of both distributions are unknown and non-common. In Section 3, we carry out a similar inference, made in the previous section, about R when α and λ are common unknown. MLE of the Stress-Strength model and non-informative BE of R are studied when α and λ are fixed in section 4. Simulation results will be studied in Sections 5. Illustrate an application of the above study in real life application in section 6.

2 Estimation of R in the General Case

In this section, the Stress-Strength model $R = \mathrm{P}(\mathrm{Y} < X)$ will be estimated, when X ~ WL distribution $(\alpha_1,\ \lambda_1,\ \beta_1)$ and Y ~WL distribution $(\alpha_2,\ \lambda_2,\ \beta_2)$. We present the MLE and non-informative BE are studied.

2.1 Maximum likelihood estimation of R in the general case

Suppose further $(\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_\mathrm{n})$ is a random sample from WL distribution $(\alpha_1,\ \lambda_1,\ \beta_1)$ and $(\mathrm{Y}_1, \mathrm{Y}_2, \dots, \mathrm{Y}_\mathrm{m})$ is another random sample from WL distribution $(\alpha_2,\ \lambda_2,\ \beta_2)$. We can find the Stress-strength parameter as equation (3).

The log-likelihood function of the observed samples is presented as:

$$\mathcal{L} = n\, ln[\Gamma(\alpha_1+1)] + n(\alpha_1-\beta_1+1)\, ln[\lambda_1] - n\, ln[\Gamma(\alpha_1-\beta_1+1)] - n\, ln[\Gamma(\beta_1)] + (\beta_1-1)\sum_{i=1}^{n} \ln[x_i] - (\alpha_1+1)\sum_{i=1}^{n} \ln[x_i+\lambda_1] + m\, ln[\Gamma(\alpha_2+1)] + m(\alpha_2-\beta_2+1)\, ln[\lambda_2] - m\, ln[\Gamma(\alpha_2-\beta_2+1)] - m\, ln[\Gamma(\beta_2)] + (\beta_2-1)\sum_{j=1}^{m} \ln[y_j] - (\alpha_2+1)\sum_{j=1}^{m} ln[y_j+\lambda_2].$$

The estimated values of $(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2)$ denoted by $(\widehat{\alpha_1}, \widehat{\lambda_1}, \widehat{\beta_1}, \widehat{\alpha_2}, \widehat{\lambda_2}, \widehat{\beta_2})$ can be derived as follows:

$$\frac{n(1+\alpha_1-\beta_1)}{\lambda_1} - (1+\alpha_1)\sum_{i=1}^{n}\frac{1}{x_i+\lambda_1} = 0, \tag{4}$$

$$nln[\lambda_1] + n\psi(\alpha_1+1) - n\psi(\alpha_1-\beta_1+1) - \sum_{i=1}^{n} ln[\lambda_1+x_i] = 0 \tag{5}$$

$$-nln[\lambda_1] - n\psi(\alpha_1-\beta_1+1) - n\psi(\beta_1) + \sum_{i=1}^{n} ln[x_i] = 0, \tag{6}$$

$$\frac{m(1+\alpha_2-\beta_2)}{\lambda_2} - (1+\alpha_2)\sum_{j=1}^{m}\frac{1}{y_j+\lambda_2} = 0, \tag{7}$$

$$mln[\lambda_2] + m\psi(\alpha_2+1) - m\psi(\alpha_2-\beta_2+1) - \sum_{j=1}^{m} ln[\lambda_2+y_j] = 0, \tag{8}$$

and

$$-mln[\lambda_2] - m\psi(\alpha_2-\beta_2+1) - m\psi(\beta_2) + \sum_{j=1}^{m} ln[y_j] = 0, \tag{9}$$

where $\psi(.)$ is the Digamma function.

The solution of equations (4-9) is not possible in closed form, so numerical technique is needed to solve it.

Then, the MLE of R is given by:

$\mathrm{R}_{\mathrm{ML1}} = \Big\{\Big[\frac{(\hat{\lambda}_2/\hat{\lambda}_1)^{\hat{\alpha}_2-\hat{\beta}_2+1}\Gamma(\hat{\alpha}_2+1)\Gamma(2+\hat{\alpha}_1+\hat{\alpha}_2-\hat{\beta}_1-\hat{\beta}_2)\Gamma(-1-\hat{\alpha}_2+\hat{\beta}_1+\hat{\beta}_2)}{(\alpha_2-\beta_2+1)\Gamma(\beta_1)\Gamma(\beta_2)(\alpha_1-\beta_1+1)\Gamma(\alpha_2-\beta_2+1)} \times\ {}_{\mathrm{p}}\mathrm{F}_{\mathrm{q}}(\{1+\hat{\alpha}_2, 1+\hat{\alpha}_2-\hat{\beta}_2, 2+\hat{\alpha}_1+\hat{\alpha}_2-$
β1−β2},{2+α2−β2,2+α2−β1−β2};λ2λ1)+(λ2/λ1)β1Γα1+1Γ1+α2−β1−β2Γβ1+β2(α2−β2+1)Γβ1Γβ2α
1−β1+1Γα2−β2+1× pFq(1+α1,β1,β1+β2,1+β1,−α2+β1+β2;λ2λ1)
(10)

2.2 Bayes estimation of R in the general case

The non-informative prior information of $(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2)$ according to Sinha [8] is given by:

$$f(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2) \propto \left(\frac{1}{\alpha_1 \cdot \lambda_1 \cdot \beta_1 \cdot \alpha_2 \cdot \lambda_2 \cdot \beta_2}\right);\ x \geq 0\ , \lambda_{1,2} > 0, \alpha_{1,2} > 0, 0 < \beta_1 < \alpha_1 + 1, 0 < \beta_2 < \alpha_2 + 1 \tag{11}$$

Combining the joint prior density of $(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2)$ and the likelihood function to obtain the joint posterior density of $(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2)$ as the form:

$$\pi\,(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2|X, Y) = \frac{f(\alpha_1,\lambda_1,\beta_1,\alpha_2,\lambda_2,\beta_2)\ f\ (x,y|\alpha_1,\lambda_1,\beta_1,\alpha_2,\lambda_2,\beta_2)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(\alpha_1,\lambda_1,\beta_1,\alpha_2,\lambda_2,\beta_2)\ f\ (x,y|\alpha_1,\lambda_1,\beta_1,\alpha_2,\lambda_2,\beta_2)\ d\alpha_1\ d\lambda_1 d\beta_1 d\alpha_2\ d\lambda_2\ d\beta_2}, \tag{12}$$

$$\pi(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2|X, Y) = \frac{k^{-1}}{\alpha_1.\ \lambda_1.\ \beta_1.\alpha_2.\ \lambda_2.\ \beta_2}\{n\ ln[\Gamma(\alpha_1+1)] + n(\alpha_1-\beta_1+1)\ ln[\lambda_1] - n\ ln[\Gamma(\alpha_1 -$$
$$\beta1+1-n \quad ln\Gamma\beta1+\beta1-1i=1n\ln xi-\alpha1+1i=1n\ln xi+\lambda1+m \quad ln\Gamma\alpha2+1+m\alpha2-\beta2+1 \quad ln\lambda2-m$$
$$ln\Gamma\alpha2-\beta2+1-m\ ln\Gamma\beta2+\beta2-1j=1m\ln yj-(\alpha2+1)j=1mln[yj+\lambda2], \tag{13}$$

Where
$$k = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2)\ f\ (x, y|\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2)\ d\alpha_1\ d\lambda_1 d\beta_1 d\alpha_2\ d\lambda_2\ d\beta_2,$$

Therefore, the Bayesian estimator of R under squared error loss function is given by:

$$\hat{R}_{BS1} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R\ \pi\,(\alpha_1,\ \lambda_1,\ \beta_1, \alpha_2,\ \lambda_2,\ \beta_2|X, Y)\ d\alpha_1\ d\lambda_1 d\beta_1 d\alpha_2\ d\lambda_2\ d\beta_2 \tag{14}$$

The Bayes estimate of R under squared error loss cannot be computed analytically. Alternatively, numerical solution based on MATHCAD15 program is employed to evaluate $\hat{R}_{BS1}$ for different values of the parameters.

3 Estimation of R with Common Unknown Parameters (α, λ)

In this section, the Stress-Strength model $R = P\,(Y < X)$ will be estimated, where the parameters $(\alpha_1 = \alpha_2 = \alpha)$ and $(\lambda_1 = \lambda_2 = \lambda)$ when X ~WL distribution$(\alpha, \lambda,\ \beta_1)$ and Y ~WL distribution$(\alpha, \lambda,\ \beta_2)$. The MLE and non-informative BE are studied. The stress-strength parameter, as equation (3), then R can be written as:

$$R = \left\{\left[\frac{\Gamma(\alpha+1)\Gamma(2+\alpha+\alpha-\beta_1-\beta_2)\Gamma(-1-\alpha+\beta_1+\beta_2)}{(\alpha-\beta_2+1)\Gamma(\beta_1)\Gamma(\beta_2)(\alpha-\beta_1+1)\Gamma(\alpha-\beta_2+1)} \times\ {}_pF_q(\{1+\alpha, 1+\alpha-\beta_2, 2+2\alpha-\beta_1-\beta_2\}, \{2+\alpha-\beta_2, 2+\alpha -\right.\right.$$
$$\beta1-\beta2\};1)+\Gamma\alpha+1\Gamma1+\alpha-\beta1-\beta2\Gamma\beta1+\beta2(\alpha-\beta2+1)\Gamma\beta1\Gamma\beta2\alpha-\beta1+1\Gamma\alpha-\beta2+1\times$$
$$pFq(1+\alpha,\beta1,\beta1+\beta2,1+\beta1,-\alpha+\beta1+\beta2;1), \tag{15}$$

3.1 MLE of R with common unknown parameters (α, λ)

Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ are random samples of n and m units, which are observed from the weighted Lomax distribution. The MLE of R is denoted by $\hat{R}_{ML2}$. To compute the MLE of R, the corresponding log-likelihood of the observed sample is given by:

$$\mathcal{L} = n\ ln[\Gamma(\alpha+1)] + n(\alpha-\beta_1+1)\ ln[\lambda] - n\ ln[\Gamma(\alpha-\beta_1+1)] - n\ ln[\Gamma(\beta_1)] + (\beta_1-1)\sum_{i=1}^{n}\ln[x_i] -$$
$$(\alpha+1)\sum_{i=1}^{n}\ln[x_i+\lambda] + m\ ln[\Gamma(\alpha+1)] + m(\alpha-\beta_2+1)\ ln[\lambda] - m\ ln[\Gamma(\alpha-\beta_2+1)] - m\ ln[\Gamma(\beta_2)] +$$
$$(\beta_2-1)\sum_{j=1}^{m}\ln[y_j] - (\alpha+1)\sum_{j=1}^{m} ln[y_j+\lambda].$$

The MLE of $(\alpha, \lambda, \beta_1, \beta_2)$ denoted by $(\hat{\alpha}, \hat{\lambda}, \widehat{\beta_1}, \widehat{\beta_2})$ can be derived by solving the following equations:

$$\frac{n(1+\alpha-\beta_1)}{\lambda} - (1+\alpha)\sum_{i=1}^{n}\frac{1}{x_i+\lambda} + \frac{m(1+\alpha-\beta_2)}{\lambda} - (1+\alpha)\sum_{j=1}^{m}\frac{1}{y_j+\lambda} = 0, \tag{16}$$

$$nln[\lambda] + n\psi(\alpha+1) - n\psi(\alpha-\beta_1+1) - \sum_{i=1}^{n} ln[\lambda+x_i] + mln[\lambda] + m\psi(\alpha+1) - m\psi(\alpha-\beta_2+1) -$$
$$\sum_{j=1}^{m} ln[\lambda+y_j] = 0 \tag{17}$$

$$-nln[\lambda] - n\psi(\alpha-\beta_1+1) - n\psi(\beta_1) + \sum_{i=1}^{n} ln[x_i] = 0, \tag{18}$$

and

$$-mln[\lambda] - n\psi(\alpha - \beta_2 + 1) - m\psi(\beta_2) + \sum_{j=1}^{m} ln[y_j] = 0, \tag{19}$$

From the non-linear equations (18) and (19), we can obtain the estimated values of β_1 and β_2 as function of α and λ by replacing $\widehat{\beta_1}, \widehat{\beta_2}$ in equations (16) and (17). The estimated values of α and λ can be then achieved.

Finally, the MLE of R, which denoted by $\hat{R}_{ML2}$, can be given by:

$\mathrm{R_{ML2}} = \left\{\left[\frac{\Gamma(\hat{\alpha}+1)\Gamma(2+2\hat{\alpha}-\hat{\beta}_1-\hat{\beta}_2)\Gamma(-1-\hat{\alpha}+\hat{\beta}_1+\hat{\beta}_2)}{(\hat{\alpha}-\hat{\beta}_2+1)\Gamma(\hat{\beta}_1)\Gamma(\hat{\beta}_2)(\hat{\alpha}-\hat{\beta}_1+1)\Gamma(\hat{\alpha}-\hat{\beta}_2+1)} \times\ {}_p\mathrm{F}_q(\{1+\hat{\alpha}, 1+\hat{\alpha}-\hat{\beta}_2, 2+2\hat{\alpha}-\hat{\beta}_1-\beta_2\}, \{2+\hat{\alpha}-\hat{\beta}_2, 2+\right.\right.$
α−β1−β2};1)+Γα+1Γ1+α−β1−β2Γβ1+β2(α−β2+1)Γβ1Γβ2α−β1+1Γα−β2+1× pFq(1+α, β1, β1+β2,1+β1,−α+β1+β2;1)
(20)

3.2 Bayes estimation of R with common unknown parameters $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$

The non-informative prior information of $(\alpha,\ \lambda,\ \beta_1,\ \beta_2)$ is:

$$f(\alpha,\ \lambda,\ \beta_1,\ \beta_2) \propto \left(\frac{1}{\alpha.\lambda.\beta_1.\beta_2}\right);\ x \geq 0\ , \lambda\ > 0, \alpha > 0, 0 < \beta_1 < \alpha_1 + 1, 0 < \beta_2 < \alpha_2 + 1 \tag{21}$$

Combining the joint prior density of $(\alpha,\ \lambda,\ \beta_1,\ \beta_2)$ and the likelihood function to obtain the joint posterior density of $(\alpha,\ \lambda,\ \beta_1,\ \beta_2)$ as the form:

$$\pi\,(\alpha,\ \lambda,\ \beta_1,\ \beta_2|X,Y) = \frac{f(\alpha,\lambda,\beta_1,\beta_2)\ f\ (x,y|\alpha,\lambda,\beta_1,\beta_2)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(\alpha,\lambda,\beta_1,\beta_2)\ f\ (x,y|\alpha,\lambda,\beta_1,\beta_2)\ d\alpha\ d\lambda d\beta_1\ d\beta_2}, \tag{22}$$

$\pi(\alpha,\ \lambda,\ \beta_1,\ \beta_2|X,Y) = \frac{k^{-1}}{\alpha.\lambda.\beta_1.\beta_2}\{n\ ln[\Gamma(\alpha+1)] + n(\alpha-\beta_1+1)\ ln[\lambda] - n\ ln[\Gamma(\alpha-\beta_1+1)] -$
*n lnΓβ1+β1−1i=1n*ln*xi−α+1i=1n*ln*xi+λ+m ln*Γ*α+1+mα−β2+1 lnλ−m ln*Γ*α−β2+1−m ln*Γ*β2+β2−1j=1m*ln*yj−(α+1)j=1mln[yj+λ],* (23)

Where $k = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(\alpha,\ \lambda,\ \beta_1,\ \beta_2)\ f\ (x,y|\alpha,\ \lambda,\ \beta_1,\ \beta_2)\ d\alpha\ d\lambda d\beta_1\ d\beta_2,$

Therefore, the Bayesian estimator of R under squared error loss function is given by:

$$\hat{R}_{BS2} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R\ \pi\,(\alpha,\lambda,\beta_1,\beta_2|X,Y)\ d\alpha\ d\lambda\ d\beta_1\ d\beta_2$$

The Bayes estimate of R under squared error loss cannot be computed analytically. Alternatively, numerical solution based on MATHCAD15 program is employed to evaluate $\hat{R}_{BS2}$ for different values of the parameters.

4 Estimation of R with Fixed Parameters $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$

In this section, the Stress-Strength model $R = \mathrm{P}\,(\mathrm{Y} < X)$ will be estimated, where the parameters $(\alpha_1 = \alpha_2 = \alpha = 1)$ and $(\lambda_1 = \lambda_2 = 2)$ when X ~WL distribution$(1, 2,\ \beta_1)$ and Y ~WL distribution$(1, 2,\ \beta_2)$. The MLE and non-informative BE are studied. The stress-strength parameter, as equation (3), then R can be written as:

$$\mathrm{R} = \left\{\left[\frac{\Gamma(2)\Gamma(4-\beta_1-\beta_2)\Gamma(-2+\beta_1+\beta_2)}{(2-\beta_2)\Gamma(\beta_1)\Gamma(\beta_2)(2-\beta_1)\Gamma(2-\beta_2)} \times\ {}_p\mathrm{F}_q(\{2,\ 2-\beta_2,\ 4-\beta_1-\beta_2\}, \{3-\beta_2, 3-\beta_1-\beta_2\}; 1)\right] + \left[\frac{\Gamma(2)\Gamma(2-\beta_1-\beta_2)\Gamma(\beta_1+\beta_2)}{(2-\beta_2)\Gamma(\beta_1)\Gamma(\beta_2)(2-\beta_1)\Gamma(2-\beta_2)} \times\ {}_p\mathrm{F}_q(\{2,\ \beta_1,\ \beta_1+\beta_2\}, \{1+\beta_1, \beta_1+\beta_2-1\}; 1)\right]\right\}, \tag{24}$$

4.1 MLE of R with fixed parameters (α, λ)

Suppose that $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ are random samples of n and m units, which are observed from the WL distribution. The MLE of R is denoted by $\widehat{R}_{ML3}$. To compute the MLE of R, the corresponding log-likelihood of the observed sample is given by:

$\mathcal{L} = n\, ln[\Gamma(2)] + n(2-\beta_1)\, ln[2\ \ ] - n\, ln[\Gamma(2-\beta_1)] - n\, ln[\Gamma(\beta_1)] + (\beta_1 - 1)\sum_{i=1}^{n} \ln[x_i] - 2\sum_{i=1}^{n} \ln[x_i +$
2+m lnΓ2+m1−β2+1 ln2−m lnΓ2−β2−m lnΓβ2+β2−1j=1mlnyj−2j=1mln[yj+2].

The MLE of (β_1, β_2) denoted by $(\widehat{\beta_1}, \widehat{\beta_2})$ can be derived by solving the following equations:

$$-nln[2] - n\psi(2-\beta_1) - n\psi(\beta_1) + \sum_{i=1}^{n} ln[x_i] = 0, \quad (25)$$

and

$$-mln[2] - n\psi(2-\beta_2) - m\psi(\beta_2) + \sum_{j=1}^{m} ln[y_j] = 0, \quad (26)$$

From the non-linear equations (24) and (25), we can obtain the estimated values of β_1 and β_2.

Finally, the MLE of R, which denoted by $\widehat{R}_{ML3}$, can be given by:

$$R_{ML2} = \left\{\left[\frac{\Gamma(2)\Gamma(4-\hat\beta_1-\hat\beta_2)\Gamma(\hat\beta_1+\hat\beta_2-2)}{(2-\hat\beta_2)\Gamma(\hat\beta_1)\Gamma(\hat\beta_2)(2-\hat\beta_1)\Gamma(2-\hat\beta_2)} \times\ {}_pF_q(\{2, 2-\hat\beta_2,\ 4-\hat\beta_1-\beta_2\}, \{3-\hat\beta_2, 3-\hat\beta_1-\hat\beta_2\}; 1)\right] + \left[\frac{\Gamma(2)\Gamma(2-\hat\beta_1-\hat\beta_2)\Gamma(\hat\beta_1+\hat\beta_2)}{(2-\beta_2)\Gamma(\beta_1)\Gamma(\beta_2)(2-\beta_1)\Gamma(2-\beta_2)} \times\ {}_pF_q(\{2,\ \hat\beta_1,\ \hat\beta_1+\hat\beta_2\}, \{1+\hat\beta_1, \hat\beta_1+\hat\beta_2-1\}; 1)\right]\right\} \quad (27)$$

4.2 Bayes estimation of R with fixed parameters (α, λ)

The non-informative prior information of $(1, 2,\ \beta_1,\ \beta_2)$ is:

$$f(1,\ 2,\ \beta_1,\ \beta_2) \propto \left(\frac{1}{2.\beta_1.\beta_2}\right);\ x \geq 0\ ,\ 0 < \beta_1 < 2, 0 < \beta_2 < 2 \quad (28)$$

Combining the joint prior density of $(1,\ 2,\ \beta_1,\ \beta_2)$ and the likelihood function to obtain the joint posterior density of $(1,\ 2,\ \beta_1,\ \beta_2)$ as the form:

$$\pi\,(1,\ 2,\ \beta_1,\ \beta_2|X,Y) = \frac{f(1,2,\beta_1,\beta_2)\ f\ (x,y|1,2,\beta_1,\beta_2)}{\int_0^\infty \int_0^\infty f(1,2,\beta_1,\beta_2)\ f\ (x,y|1,2,\beta_1,\beta_2)\ \beta_1\ d\beta_2}, \quad (29)$$

$\pi(\alpha,\ \lambda,\ \beta_1,\ \beta_2|X,Y) = \frac{k^{-1}}{2.\beta_1.\beta_2}\{n\, ln[\Gamma(2)] + n(2-\beta_1)\, ln[2\ \ ] - n\, ln[\Gamma(2-\beta_1)] - n\, ln[\Gamma(\beta_1)] +$
β1−1i=1nlnxi−2i=1nlnxi+2+m lnΓ2+m1−β2+1 ln2−m lnΓ2−β2−m lnΓβ2+β2−1j=1mlnyj−2j=1mln[yj+2], (30)

Where $k = \int_0^\infty \int_0^\infty f(1,\ 2,\ \beta_1,\ \beta_2)\, f\ (x,y|1,\ 2,\ \beta_1,\ \beta_2)\ \beta_1\ d\beta_2$,

Therefore, the Bayesian estimator of R under squared error loss function is given by:

$$\hat{R}_{BS2} = \int_0^\infty \int_0^\infty R\ \pi\ (1,2,\beta_1,\beta_2|X,Y)\ \ d\beta_1\ d\beta_2 \quad (31)$$

The Bayes estimate of R under squared error loss cannot be computed analytically. Alternatively, numerical solution based on MATHCAD15 program is employed to evaluate $\hat{R}_{BS2}$ for different values of the parameters.

5 Simulation Results

In this section, Monte Carlo simulation is performed to test the behavior of the proposed estimators for different sample sizes and for different parameter values.

The Performance of the maximum likelihood estimates and the Bayes estimates are compared in terms of biases and mean squares errors (MSEs). (BE) are computed based non-informative prior distribution, where we have three cases for the weighted Lomax distribution.

Case 1: When the parameters are unknown and different for X and Y.
Case 2: When the parameters (α, λ) are common unknown where $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_1 = \lambda_2 = \lambda$ for X and Y, respectively.
Case 3: When the parameters (α, λ) are fixed where $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_1 = \lambda_2 = \lambda$ for X and Y , respectively.

MLE of the unknown parameters of the weighted Lomax distribution and the Bayes estimators of the reliability function of the weighted Lomax distribution will be obtained by the same way. The following steps will be considered to obtain the estimators:

After generating random samples $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ from weighted Lomax distribution with small, medium and large sample sizes then we have three cases:

Case1, where the parameters are unknown and different $(\alpha_1, \lambda_1, \beta_1, \alpha_2, \lambda_2, \beta_2)$ are unknown for X, Y. So, we take different initial values such as:

$$\alpha_1 = 2;\ \lambda_1 = 0.3\ ;\ \beta_1 = 2;\ \alpha_2 = 1.8;\ \lambda_2 = 0.5\ ;\ \beta_2 = 1$$
$$\alpha_1 = 2.3;\ \lambda_1 = 2\ ;\ \beta_1 = 1.5;\ \alpha_2 = 2.1;\ \lambda_2 = 2.5\ ;\ \beta_2 = 1$$

Their results for MLE and BE shown in Tables 1 and 2.

Table 1. Monte Carlo simulation results: Bias and MSE for MLE and Bayesian at $\boldsymbol{\alpha_1 = 2;\ \lambda_1 = 0.3\ ;\ \beta_1 = 2;\ \alpha_2 = 1.8; \lambda_2 = 0.5\ ; \beta_2 = 1}$

			$\boldsymbol{\alpha_1 = 2;\ \lambda_1 = 0.3\ ;\ \beta_1 = 2;\ \alpha_2 = 1.8; \lambda_2 = 0.5\ ; \beta_2 = 1}$						
$\boldsymbol{n, m}$			$\boldsymbol{\alpha_1}$	$\boldsymbol{\lambda_1}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\alpha_2}$	$\boldsymbol{\lambda_2}$	$\boldsymbol{\beta_2}$	**R**
30, 30	MLE	Bias	0.4549	0.5137	0.1273	0.6957	0.7389	0.1384	-0.0048
		MSE	1.6552	4.0457	0.6518	1.7259	0.7053	0.1293	0.0038
	Bayes	Bias	0.4004	0.0496	0.0707	0.4010	0.0882	0.0269	0.0060
		MSE	0.0798	0.0526	0.0253	0.0246	0.0294	0.0169	0.0026
30, 40	MLE	Bias	-0.3390	0.4322	-0.5042	0.5808	0.6532	-0.0547	-0.0032
		MSE	0.9197	0.4333	0.3358	1.4640	0.7042	0.1095	0.0018
	Bayes	Bias	0.0426	0.0336	0.0223	0.0433	0.0580	0.0176	-0.0034
		MSE	0.0223	0.0072	0.0076	0.0233	0.0198	0.0115	0.0018
40, 50	MLE	Bias	0.2735	0.1717	-0.1599	0.4575	0.3415	0.0109	0.0027
		MSE	0.6770	1.1662	0.4467	1.1397	0.5789	0.1063	0.0003
	Bayes	Bias	0.0125	0.0154	0.0584	-0.0179	0.0386	-0.0010	0.0026
		MSE	0.0169	0.0248	0.0250	0.0232	0.0192	0.0064	0.0009
60, 60	MLE	Bias	0.1301	0.1376	0.1750	0.4120	0.2848	0.0091	-0.0018
		MSE	0.4404	0.1605	0.4380	0.9669	0.3114	0.0932	0.0001
	Bayes	Bias	0.0062	0.0102	0.0450	0.0148	-0.0032	0.0009	-0.0011
		MSE	0.0029	0.0131	0.0223	0.0226	0.0181	0.0056	0.0005
70, 80	MLE	Bias	0.0361	0.1035	-0.0820	0.3750	0.1846	0.0060	-0.0009
		MSE	0.2064	0.1034	0.3184	0.8536	0.2310	0.0180	0.00009
	Bayes	Bias	0.0024	-0.0084	0.0901	0.0084	0.00136	0.0008	-0.0005
		MSE	0.0001	0.0053	0.0224	0.0207	0.0169	0.0015	0.0004
80, 80	MLE	Bias	0.0138	0.0981	0.0598	0.2003	0.1298	0.0029	-0.0004
		MSE	0.1753	0.1340	0.3191	0.6431	0.2231	0.0046	0.00002
	Bayes	Bias	0.0009	0.0074	0.0536	-0.0001	0.00065	-0.0001	0.0004
		MSE	0.00006	0.0051	0.0211	0.0201	0.0002	0.0006	0.0001

Table 2. Monte Carlo simulation results: Bias and MSE for MLE and Bayesian at $\alpha_1 = 2.3$; $\lambda_1 = 2$; $\beta_1 = 1.5$; $\alpha_2 = 2.1$; $\lambda_2 = 2.5$; $\beta_2 = 1$

			$\alpha_1 = 2.3$; $\lambda_1 = 2$; $\beta_1 = 1.5$; $\alpha_2 = 2.1$; $\lambda_2 = 2.5$; $\beta_2 = 1$						
n, m			α_1	λ_1	β_1	α_2	λ_2	β_2	R
30, 30	MLE	Bias	1.0386	1.2253	1.3542	1.1219	0.8514	0.7477	0.0630
		MSE	3.1140	6.6501	1.0852	4.2952	5.8784	0.8754	0.0097
	Bayes	Bias	1.0035	1.1779	0.8451	0.6234	0.8086	0.1643	0.0373
		MSE	3.1687	4.1634	0.8648	2.4257	1.8428	0.1475	0.0091
30, 40	MLE	Bias	1.0186	1.0317	0.9240	0.8676	0.5938	0.5335	0.0184
		MSE	2.8840	4.6829	0.9551	2.1554	2.8033	0.4343	0.0061
	Bayes	Bias	1.0015	0.9161	0.6482	0.4158	0.4613	0.0649	0.0126
		MSE	2.0405	2.8979	0.6154	1.8458	1.6241	0.0914	0.0043
40, 50	MLE	Bias	0.0143	0.9591	0.6623	0.7183	0.4539	0.2526	0.0089
		MSE	1.5836	4.0761	0.6153	2.0222	3.8324	0.3637	0.0055
	Bayes	Bias	1.0008	0.8328	0.4008	0.3333	0.5568	0.3079	0.0084
		MSE	1.5181	2.0339	0.5208	1.1192	1.7430	0.0191	0.0015
60, 60	MLE	Bias	0.0120	0.6155	0.1975	0.6574	0.3304	0.1366	0.0067
		MSE	0.8195	2.5472	0.3568	1.5531	2.9668	0.2770	0.0039
	Bayes	Bias	0.8815	0.7000	0.2309	0.1114	0.4597	0.1222	0.0014
		MSE	0.7223	1.9469	0.3535	0.8932	1.1944	0.0083	0.0009
70, 80	MLE	Bias	0.0119	0.3893	0.0908	0.3031	0.6203	0.0626	0.0028
		MSE	0.5855	2.3375	0.2413	0.6449	2.6183	0.0933	0.0022
	Bayes	Bias	0.3731	0.5049	0.2090	0.0768	0.6639	0.0514	0.0006
		MSE	0.4064	1.4057	0.1483	0.4981	1.0748	0.0022	0.0004
80, 80	MLE	Bias	0.0056	0.2230	0.011	0.2014	0.3141	0.0201	-0.0075
		MSE	0.6214	1.6915	0.1074	0.4445	2.2584	0.0115	0.0019
	Bayes	Bias	0.1077	0.0079	0.1743	0.0126	0.3213	0.0137	0.0001
		MSE	0.1394	0.0944	0.0561	0.3429	0.2052	0.0008	0.0002

Case 2, the parameters (α, λ) are common unknown also, the parameters β_1, β_2 are unknown for X, Y, respectively where the initial values can be taken as follow:

$$\alpha_1 = \alpha_2 = 1.5, \quad \lambda_1 = \lambda_2 = 0.9, \quad \beta_1 = 1.2, \quad \beta_2 = 2.15$$
$$\alpha_1 = \alpha_2 = 2.2, \quad \lambda_1 = \lambda_2 = 1.2, \quad \beta_1 = 1.5, \quad \beta_2 = 0.95$$

Their results for MLE and BE shown in Tables 3 and 4.

Case 3, the parameters (α, λ) are common and fixed where $\alpha = 1$ and $\lambda = 2$, where parameters

$$\alpha_1 = \alpha_2 = 1, \quad \lambda_1 = \lambda_2 = 2, \quad \beta_1 = 2.2, \quad \beta_2 = 1.6$$
$$\alpha_1 = \alpha_2 = 1, \quad \lambda_1 = \lambda_2 = 2, \quad \beta_1 = 1.7, \quad \beta_2 = 2.2$$

Their results shown in Tables 5 and 6.

Table 3. Monte Carlo simulation results: Bias and MSE for MLE and Bayesian at $\alpha_1 = \alpha_2 = 1.5$; $\lambda_1 = \lambda_2 = 0.9$; $\beta_1 = 1.2$; $\beta_2 = 2.15$

			$\alpha_1 = \alpha_2 = 1.5$; $\lambda_1 = \lambda_2 = 0.9$; $\beta_1 = 1.2$; $\beta_2 = 2.15$						
n, m			α_1	λ_1	β_1	α_2	λ_2	β_2	R
30, 30	MLE	Bias	0.658	0.721	0.625	0.875	0.758	0.557	0.8325
		MSE	1.985	2.004	0.875	1.875	1.697	0.947	0.6086
	Bayes	Bias	0.3578	0.5087	0.3168	0.2589	0.2085	0.4028	0.5247
		MSE	0.8452	0.4208	0.4049	0.3325	0.5002	0.4015	0.2041
30, 40	MLE	Bias	0.4145	0.5248	0.4158	0.6257	0.6145	0.2428	0.5121
		MSE	0.7152	1.5124	0.5487	1.5870	0.9852	0.6045	0.5241
	Bayes	Bias	0.1524	0.3458	0.2125	0.1793	0.1403	0.2823	0.3029
		MSE	0.6425	0.2102	0.2023	0.2486	0.3058	0.2283	0.1104

40, 50	MLE	Bias	0.3102	0.3251	0.3102	0.5843	0.4121	0.1804	0.1903
		MSE	0.5152	0.9021	0.4231	2.0014	0.6233	0.4121	0.3134
	Bayes	Bias	0.0852	0.2146	0.1178	0.1289	0.1103	0.1012	0.1121
		MSE	0.3025	0.1147	0.1839	0.1948	0.1891	0.2043	0.1025
60, 60	MLE	Bias	0.1124	0.1345	0.3089	0.5249	0.2361	0.1581	0.0832
		MSE	0.3214	0.6103	0.3201	0.9941	0.4257	0.2657	0.1058
	Bayes	Bias	0.0313	0.1047	0.0825	0.1158	0.0158	0.0859	0.0982
		MSE	0.1025	0.0921	0.1329	0.1403	0.1077	0.1543	0.0379
70, 80	MLE	Bias	0.0095	0.1088	0.1149	0.3542	0.2014	0.1122	0.0335
		MSE	0.1851	0.2017	0.1452	0.9014	0.3951	0.1088	0.0884
	Bayes	Bias	0.0128	0.1095	0.0233	0.1047	0.0262	0.0328	0.0291
		MSE	0.0428	0.0363	0.1166	0.1063	0.1005	0.1022	0.0058
80, 80	MLE	Bias	0.0038	0.0074	0.1033	0.3321	0.1067	0.0479	0.0128
		MSE	0.0522	0.4489	0.1183	0.7254	0.3698	0.0851	0.0313
	Bayes	Bias	0.0087	0.1074	0.0106	0.0105	0.0108	0.0012	0.0105
		MSE	0.0087	0.0123	0.0087	0.0157	0.0072	0.0158	0.0012

Table 4. Monte Carlo simulation results: Bias and MSE for MLE and Bayesian at $\alpha_1 = \alpha_2 = 2.2$, $\lambda_1 = \lambda_2 = 1.2$, $\beta_1 = 1.5$, $\beta_2 = 0.95$

			$\alpha_1 = \alpha_2 = 2.2$,	$\lambda_1 = \lambda_2 = 1.2$,	$\beta_1 = 1.5$,	$\beta_2 = 0.95$			
n, m			α_1	λ_1	β_1	α_2	λ_2	β_2	R
30, 30	MLE	Bias	1.245	1.4618	1.5191	1.3342	0.9925	1.3017	0.7247
		MSE	3.5287	6.9112	1.3151	4.2246	5.9921	0.9953	0.8243
	Bayes	Bias	1.7151	1.4325	0.8731	0.9451	1.0025	0.8213	0.3389
		MSE	3.0182	4.5101	0.9103	1.6658	2.0051	0.4369	0.7515
30, 40	MLE	Bias	1.1458	1.2088	0.9394	0.7252	0.8804	0.8403	0.2457
		MSE	2.9101	5.7921	1.1682	2.8523	3.9123	0.6321	0.7948
	Bayes	Bias	1.5213	1.2355	0.6103	0.7234	0.6633	0.6538	0.2911
		MSE	1.7581	3.9104	0.5531	0.9121	1.8122	0.3028	0.5102
40, 50	MLE	Bias	0.9014	0.8821	0.5327	0.6023	0.7231	0.6361	0.1157
		MSE	2.6422	4.2315	1.0325	2.1328	2.9943	0.5433	0.5821
	Bayes	Bias	1.3325	1.0478	0.3325	0.5426	0.8932	0.3017	0.1691
		MSE	1.4325	2.4587	0.4214	0.5921	1.9921	0.2022	0.4347
60, 60	MLE	Bias	0.5231	0.8256	0.3325	0.5305	0.6611	0.4123	0.1033
		MSE	2.0325	2.8921	0.9291	1.8211	2.6521	0.4825	0.4285
	Bayes	Bias	0.9732	0.9254	0.2592	0.3325	0.7211	0.2361	0.1489
		MSE	0.9921	2.1145	0.2231	0.1902	1.3212	0.1214	0.2253
70, 80	MLE	Bias	0.5123	0.6323	0.3025	0.3528	0.8273	0.3125	0.1015
		MSE	0.9325	2.5203	0.7413	0.8101	2.5232	0.4155	0.2031
	Bayes	Bias	0.4321	0.9052	0.1011	0.1232	0.8921	0.1109	0.0289
		MSE	0.6214	1.9213	0.1152	0.0328	1.9821	0.1099	0.1085
80, 80	MLE	Bias	0.6824	0.4214	0.2282	0.1421	0.5391	0.2314	0.1011
		MSE	0.8213	1.9234	0.5681	0.6602	2.2829	0.4292	0.1207
	Bayes	Bias	0.2023	0.3028	0.0691	0.0827	0.7231	0.0452	0.0179
		MSE	0.4325	0.2457	0.0898	0.0197	1.5838	0.0921	0.0182

Table 5. Monte Carlo simulation results: Bias and MSE for MLE and Bayesian at $\alpha_1 = \alpha_2 = 1$, $\lambda_1 = \lambda_2 = 2$, $\beta_1 = 1.9$, $\beta_2 = 1.6$

			$\alpha_1 = \alpha_2 = 1$,	$\lambda_1 = \lambda_2 = 2$,	$\beta_1 = 1.9$,	$\beta_2 = 1.6$			
n, m			α_1	λ_1	β_1	α_2	λ_2	β_2	R
30, 30	MLE	Bias	2.251	2.527	2.902	1.9391	2.0012	2.305	1.805
		MSE	2.4215	5.328	1.926	3.824	4.6221	2.0051	1.329
	Bayes	Bias	1.3325	1.8628	1.7823	1.3321	1.9251	1.8631	1.2241
		MSE	1.2237	3.1012	1.2231	2.441	2.9841	1.8243	0.9872
30, 40	MLE	Bias	1.8692	1.7791	1.7851	0.9981	0.9966	1.8825	1.0026
		MSE	2.1025	4.9876	1.0025	2.9521	3.7762	0.9986	0.8857
	Bayes	Bias	1.0103	1.1356	1.3103	0.9927	1.0033	1.1302	0.8897
		MSE	1.0105	2.8835	0.9851	1.1142	1.0871	0.9258	0.3312

40, 50	MLE	Bias	1.0124	1.1221	0.9921	0.5623	0.6614	1.0232	0.9821
		MSE	1.9521	3.001	0.8881	2.0821	2.0055	0.3254	0.6565
	Bayes	Bias	1.0021	1.0001	0.8989	0.6235	0.8469	0.9861	0.5521
		MSE	0.8792	1.8521	0.3325	0.9345	0.8991	0.6581	0.1008
60, 60	MLE	Bias	0.8951	0.9263	0.6617	0.3769	0.4391	0.8521	0.6154
		MSE	1.0012	2.0014	0.4321	1.6533	1.8721	0.2215	0.4582
	Bayes	Bias	0.8231	0.7251	0.6614	0.5363	0.6684	0.7253	0.3028
		MSE	0.5028	0.8631	0.0981	0.6282	1.5833	0.4022	0.0014
70, 80	MLE	Bias	0.4231	0.5369	0.3323	0.1982	0.2377	0.6301	0.4432
		MSE	0.8273	1.8321	0.2251	0.7921	1.1983	0.0982	0.2873
	Bayes	Bias	0.652	0.5103	0.3627	0.3328	0.4293	0.5396	0.0852
		MSE	0.3697	0.6212	0.0128	0.4183	1.0239	0.2236	0.0009
80, 80	MLE	Bias	0.2235	0.3128	0.2425	0.0821	0.1835	0.4756	0.3211
		MSE	0.6231	0.9875	0.0152	0.5241	0.9258	0.0196	0.1584
	Bayes	Bias	0.4528	0.2231	0.1146	0.2587	0.1983	0.3684	0.0152
		MSE	0.1083	0.4537	0.0101	0.4183	0.9231	0.0873	0.00001

Table 6. Monte Carlo simulation results: Bias and MSE for MLE and Bayesian at $\alpha_1 = \alpha_2 = 1,\ \lambda_1 = \lambda_2 = 2,\ \beta_1 = 1.7,\ \ \beta_2 = 1.6$

			$\alpha_1 = \alpha_2 = 1,$	$\lambda_1 = \lambda_2 = 2,$	$\beta_1 = 1.7,$	$\beta_2 = 1.6$			
n, m			α_1	λ_1	β_1	α_2	λ_2	β_2	**R**
30, 30	MLE	Bias	1.857	1.6621	1.8952	1.4103	1.9923	1.8765	1.0024
		MSE	1.8872	3.9852	0.8873	1.6631	1.3121	1.7216	0.9876
	Bayes	Bias	0.6352	0.9925	1.0028	0.7327	1.0001	1.2353	0.8969
		MSE	1.2325	3.0014	0.1483	1.3381	0.8812	0.9941	0.5524
30, 40	MLE	Bias	1.5239	1.2258	1.2257	1.0081	0.1214	1.3255	0.8214
		MSE	1.2251	3.0027	0.6537	0.3317	0.9957	1.1114	0.2825
	Bayes	Bias	0.4428	0.6353	0.8824	0.5284	0.9862	0.8874	0.6352
		MSE	0.9891	2.8835	0.0025	1.0251	1.6247	0.6352	0.2287
40, 50	MLE	Bias	1.2247	1.0149	0.017	0.9841	0.0817	1.1935	0.6621
		MSE	0.9821	2.587	0.4412	2.1483	0.6321	0.9924	0.0992
	Bayes	Bias	0.2215	0.3351	0.6108	0.3324	0.6633	0.5304	0.2421
		MSE	0.5357	2.3317	0.0009	0.9981	1.1187	0.3328	0.0182
60, 60	MLE	Bias	1.0983	0.9987	0.0098	0.6382	0.0157	0.9367	0.2289
		MSE	0.6321	1.8524	0.2287	1.8321	0.3972	0.7725	0.0251
	Bayes	Bias	0.1103	0.2458	0.3287	0.1188	0.2391	0.2233	0.1083
		MSE	0.3235	1.9892	0.00065	0.7259	0.9927	0.1047	0.0029
70, 80	MLE	Bias	0.9894	0.5367	0.0063	0.3587	0.0097	0.6637	0.1937
		MSE	0.4459	0.9324	0.0873	0.8934	0.1984	0.3897	0.0097
	Bayes	Bias	0.0735	0.0987	0.1079	0.0831	0.1483	0.0937	0.0015
		MSE	0.1482	1.1573	0.00022	0.4662	0.5247	0.0031	0.0004
80, 80	MLE	Bias	0.6631	0.4238	0.0028	0.1826	0.00024	0.3871	0.00257
		MSE	0.3212	0.7324	0.0127	0.3587	0.0276	0.1472	0.0014
	Bayes	Bias	0.0128	0.0257	0.0287	0.0257	0.0149	0.0157	0.0004
		MSE	0.0987	0.8896	0.000001	0.0127	0.1798	0.00098	0.00001

It can be noted that

- The MSEs of the MLE and Bayesian estimator decrease as (n, m) increase
- for large sample sizes, the performance of the BE are better than the MLE of R in terms of biases and MSEs.

6 Real Data Analysis

In this section, we used the real data sets of the waiting times before service of the customers of two banks A and B, respectively. These data sets simultaneously have been reported by Al-Mutairi et al. [9] for estimating the stress-strength reliability function. The data are as follows:

Waiting time (in minutes) before customer service in Bank A:

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8,4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9,9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7,13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1 and 38.5

Waiting time (in minutes) before customer service in Bank B:

0.1, 0.2, 0.3, 0.7, 0.9, 1.1, 1.2, 1.8, 1.9, 2.0, 2.2, 2.3, 2.3, 2.3, 2.5, 2.6, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.4, 3.5, 3.9, 4.0, 4.2,4.5, 4.7, 5.3, 5.6, 5.6, 6.2, 6.3, 6.6, 6.8, 7.3, 7.5, 7.7, 7.7, 8.0, 8.0, 8.5, 8.5, 8.7, 9.5, 10.7, 10.9, 11.0, 12.1, 12.3, 12.8, 12.9,13.2, 13.7, 14.5, 16.0, 16.5 and 28.0

Two data sets are fitted with the WL distribution. The Kolmogorov-Smirnov (K-S), and P-value are provided in Table 7. Obviously, the WL model fits well to Data Set 1 and Data Set 2. The MLE and Bayesian estimates of R for the real data are provided in Table 8. Furthermore, we also plotted the empirical cumulative distribution plot and fitted probability density function plot for both data sets, see Figs. 1, 2, 3 & 4.

Table 7. P-value and K-S statistics of different goodness-of-fit tests for data set 1, 2

	K.S	P-Value
X	0.0384	0.9985
Y	0.0624	0.9738

Table 8. Maximum likelihood ,Bayesian estimates of the parameters and R

		α_1	λ_1	β_1	α_2	λ_2	β_2	R
MLE	estimate	17.4875	66.4671	2.2632	77.1162	346.5630	1.3929	0.6290
	SE	25.8375	127.3613	0.5467	201.4592	931.8998	0.2358	
Bayesian	estimate	17.1409	66.1206	1.9166	76.7696	346.2164	1.0464	0.6977
	SE	3.2518	4.8470	0.4604	5.3056	6.8372	0.2445	

From Table 8 the performance of the Bayes estimator is better than maximum likelihood estimator.

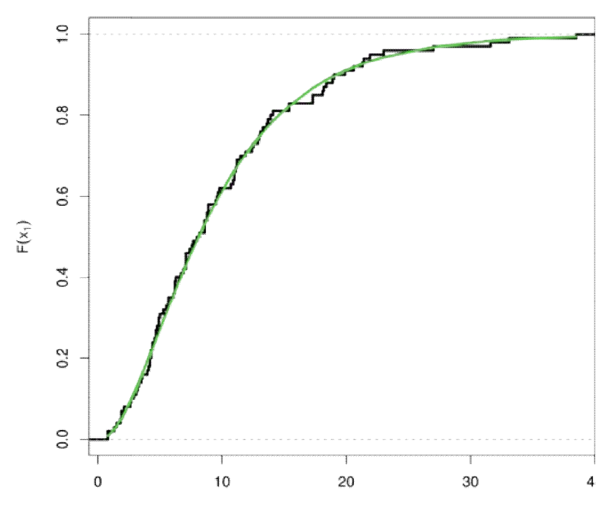

Fig. 1. Empirical cumulative distribution plot of the first data set

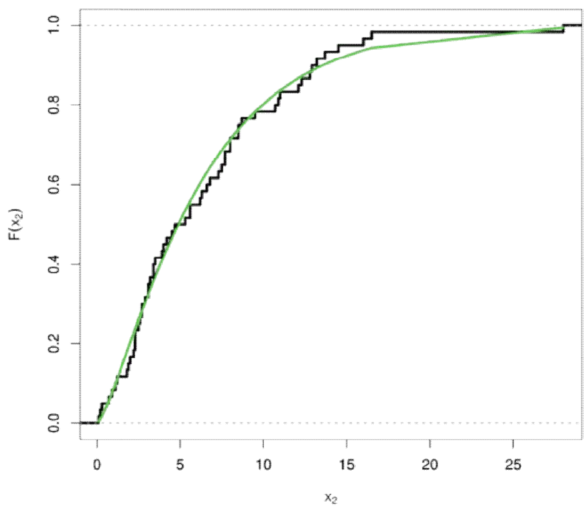

Fig. 2. Empirical cumulative distribution plot of the second data set

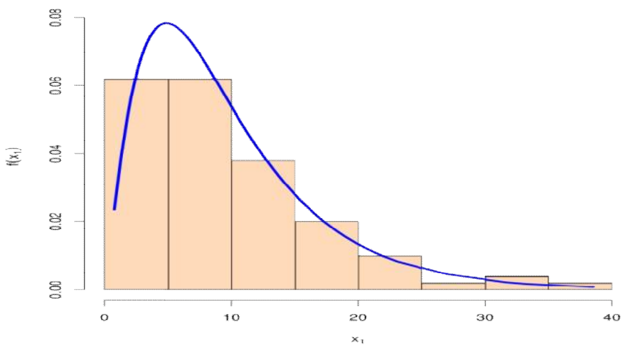

Fig. 3. Fitted probability density function plot of the first data

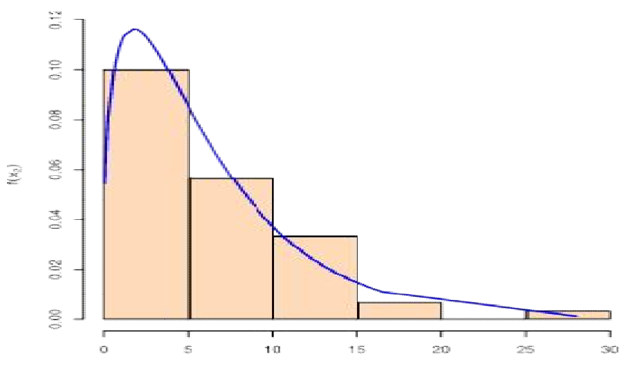

Fig. 4. Fitted probability density function plot of the second data

7 Conclusion

In this paper, we presented two methods for estimating $p(Y < X)$ when X and Y both follow WL distribution with different parameters. We investigated Maximum likelihood and Bayesian estimator methods of R and their performances are examined by the simulation study. Further, we have also considered two real data set to illustrate the applicability of WL distribution in real life and it is found that the considered model provides better fit to the given data set. Simulation results and real data application suggest that the performance of the Bayesian estimator is better than the maximum likelihood estimator for all different sample sizes; also, the maximum likelihood method provides very satisfactory results as sample size increased. It is hoped that our investigation will be useful for researchers dealing with the kind of data considered in this paper.

Competing Interests

Authors have declared that no competing interests exist.

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