

Review of Different Types of Energy and Some Properties of Semiregular Graphs

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Authors' contributions

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Abstract

In this paper, some properties of semi-regular graphs have been studied. The energy of graphs has many mathematical properties, which are being investigated for some of the semi-regular graphs. Also, the Laplacian Energy of these types of the graph has been defined has also been studied. We give examples of semi-regular graphs, describe the barbell class, and describe how the property of semi regularity relates to other properties of graphs.

Keywords: Regular graph, semi-regular, Graph energy, adjacency spectrum, Laplacian spectrum, matrices.

1. Introduction

Initially the concept of Semi regular graphs have been introduced by Balaban et al. (1972) in the form of combination graphs, and Kerek et al. (1974) analysed these types of graphs using convolution graphs. Various

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classes of semi-regular graphs described by Alison Northup [1], and also discussed an algorithm for determining a semi-regular graph from a given graph. The publication of the papers ‘Distance degree regular graphs’ by Bloom G.S and others [2], and ‘How to define an irregular graph’ by Chartr and others [3] has arisen much attention on study of the semi-regular graphs. Several matrices can be associated to a graph such as the adjacency matrix (denoted by A) or the Laplacian matrix $L = D - A$ where D is the diagonal matrix of degrees. Some structural properties can be deduced from their spectrum but in general we can’t determine a graph from its adjacency or Laplacian spectrum [4]. Ivan Niven (2018), introduced the theory of numbers using some regular graphs [5]. Merris and Grone [6-9], The Laplacian spectrum of the graph G, consisting of the numbers $\mu_1, \mu_2, \dots, \mu_n$, is the spectrum of its Laplacian matrix. Abdollah Alhevaz et al. [10], introduced the generalized sharp bounds distance matrix $D_\alpha(G) = \alpha Tr(G) (1-\alpha) D(G)$ where $\alpha \in [0, 1]$ and the graph G involving various invariants. The generalized distance on the energy graph is defined as $D_\alpha(G) = \alpha Tr(G) (1-\alpha) D(G)$ for $0 \leq \alpha \leq 1$ and if the value $\partial_1 \geq \partial_2 \geq \dots \partial_n$ are the eigenvalues the given graph, Pirzada [11]. Shaowei Sun et al. [12], introduced the properties of distance spectral radius of some clique trees. The block is referred that the maximal graph G is not cut the vertex and the order is n. The energy graph $E(G)$ is defined as

$$E(G) \leq \binom{n}{2} (1 + \sqrt{n})$$

is hold on more infinite family of graphs, Moulton, (2001)[13]. Samir et. al., (2017),

determined that the two energy graphs namely one is splitting graph as $E(S'(G)) = \sqrt{5}E(G)$ and another one is shadow graph is $E(D_2(G)) = 2E(G)$ [14]. The asymptotic behaviour of some indices of iterated line graphs of regular graphs is investigated, Liu et al. [15].

A concept related to the spectrum of a graph is that of energy. As its name suggests, it is inspired by energy in chemistry. In 1978, Gutman defined energy mathematically for all graphs [16]. In this paper, some special semi regular graphs and their properties have been discussed. Energy of these semi regular graphs are also founded.

2. Regular Graphs and Semiregular Graphs

A graph is regular if every vertex in the graph has the same degree. If all the vertices of a graph have degree n , we call that graph n -regular.

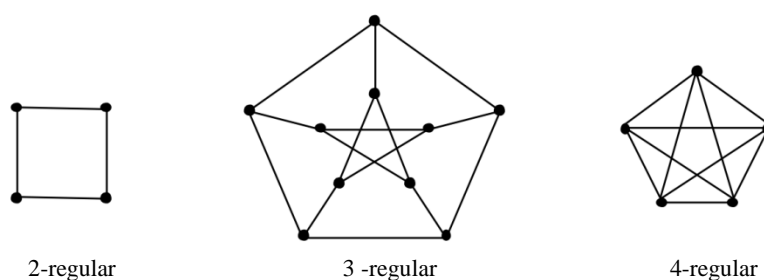


Fig. 1. Regular graph design

A simple connected graph in which each vertex is at distance 1 away from the same number of vertices is called regular graph. On the other hand, if each vertex is at distance 2 away from exactly the same number of vertices then the graph is called semi regular graph. If each vertex is at distance 2 away from exactly n -vertices, then the graph is called n -semi regular graph.

Semi regular graphs are a natural extension of the idea of regular graphs. Although extensive literature exists on regular graphs, semi-regular graphs have been much less studied.

Define $\text{deg}_2(v)$ to be the number of vertices that are distance 2 away from v in a given graph. It is obvious that the union more than one n -semi regular graph is also n -semi regular, so we will limit our discussion to connected semi regular graphs.

2.1 Example

The graphs given in Fig. 2 are examples of some semi regular graphs

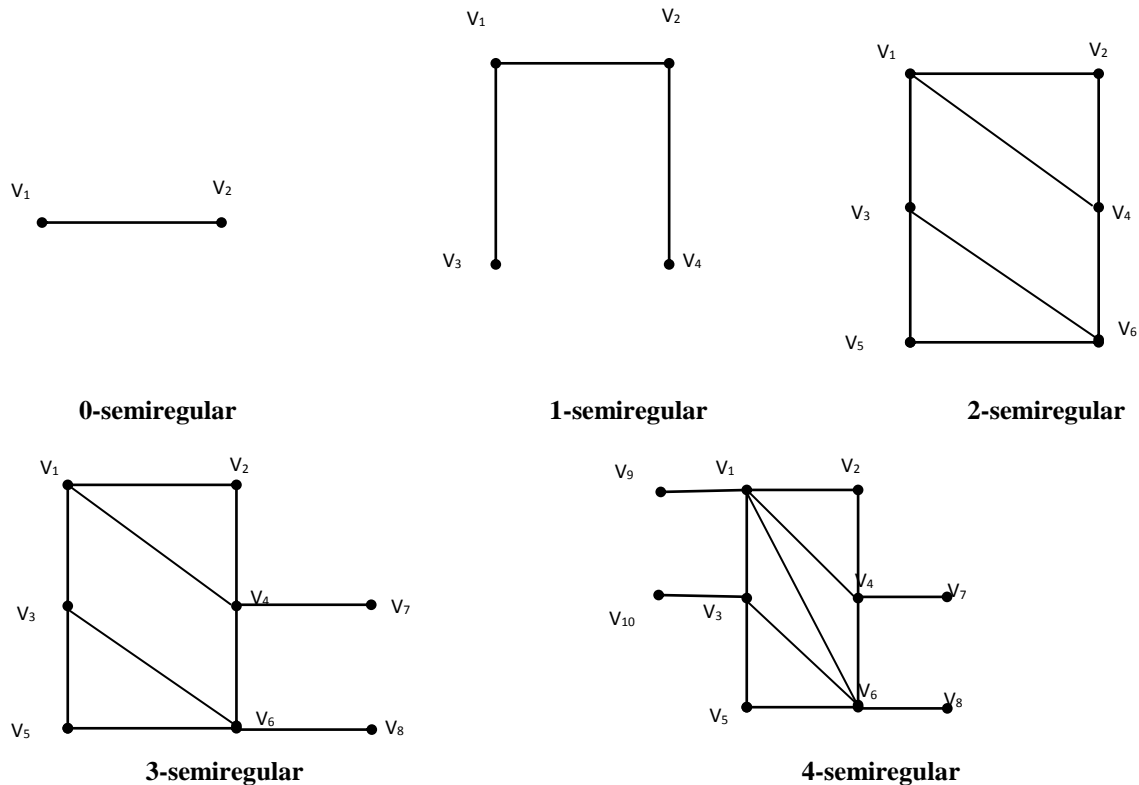


Fig. 2. semi regular graphs

From the structure of the adjacency matrices of the above semi regular graphs, it can be easily observed that they are the extensions of the 0-semiregular graph K_2 . A regular graph may not be a semi regular graph. The graph given in Fig.3 is 3-regular but not semi regular.

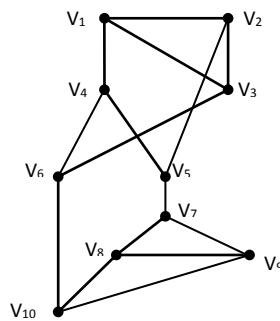


Fig. 3. Regular but not semi regular graph

2.2 Theorem

A simple connected graph is 0-semiregular if and only if it is a complete graph.

Proof:

Let G be a connected 0-semiregular graph with n vertices. The distance between any two vertices of G must be 1, because a distance greater than 1 would mean that G would have two vertices that were distance 2 apart, and

G would therefore not be 0-semiregular. A connected graph with n vertices in which all vertices are at distance 1 from all other vertices is the complete graph. Let G be the complete graph. Then for any vertex v in G , v is not distance 2 away from any other vertices. Thus, G is 0-semiregular.

Note: In the above theorem, the connectedness is necessary to prove that the 0-semiregular is complete.

2.3 Theorem

A connected graph is 1-semiregular if and only if it is P_4 or $\overline{\left(\bigcup_{i=1}^n P_2\right)}$, for $n \geq 2$.

2.4. Theorem

A connected graph is 2-semiregular if and only if it is an n -cycle or the complement of an n -cycle for $n \geq 5$, the complement of the union of at least two disjoint cycles.

2.5. Theorem

The necessary condition for a graph to be a semi regular

Let S be a semi regular graph, and let u and v be any two vertices of deg m . If there is a vertex x of deg n adjacent to u then there is a vertex y of deg n , adjacent to the vertex v .

Proof

Let S be a semi regular graph. Let $\deg u = \deg v = m$. Also let x and y are vertices adjacent to u and v respectively such that $\deg x = k$ and $\deg y = l$ where $k \neq l$. For simplicity, first let us assume that $\deg u = \deg v = 1$. Then if $\deg x \neq \deg y$, then the number of vertices which are at distance 2 for u and v will not be the same. It is a contradiction to the assumption that S is semi regular.

Next let us assume that $\deg u = \deg v > 1$, then the number of vertices which are at distance 2 to u through the vertex x is k . Similarly the number of vertices which are at distance 2 to v through y is l . Since $k \neq l$, the number of vertices which are at distance 2 from u and v are not same. Again this is a contradiction. Hence the theorem. The converse of the theorem 2.3 is not true; As an example, consider the Grotzsch's graph given in Fig.4.

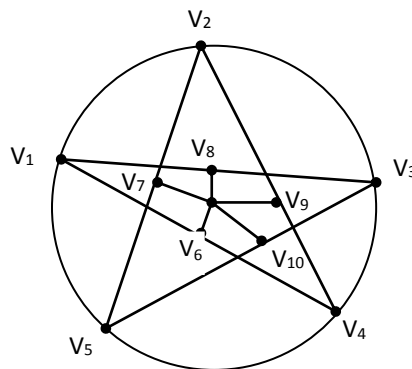


Fig. 4. Grotzsch's graph

There are vertices of deg 3, deg 4 and deg 5. Also the number of vertices of deg 3, deg 4 and deg 5 are 5, 5, 1 respectively.

Let α_{ij} denote the number adjacent vertices of deg j to the vertex of deg i . Then it can be found that $\alpha_{11}=0$; $\alpha_{12} = 2$; $\alpha_{13}= 1$; $\alpha_{21} = 2$; $\alpha_{22} = 2$; $\alpha_{23} = 0$; $\alpha_{31} = 5$; $\alpha_{32} = 0$; $\alpha_{33} = 0$.

But the above graph is not semi regular.

2.6 Theorem

If a graph S is semi regular then given any vertex u of deg m , the sum of degrees of adjacent vertices is a constant, independent of the choice of u .

Proof

Theorem 2.4 can be easily observed from the theorem 2.3

2.7 Example

As an illustration to theorem 2.4, consider the graph given in fig. 5

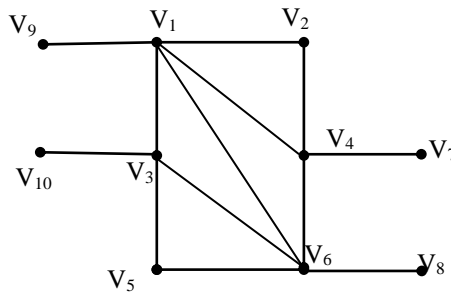


Fig. 5. 4-semiregular graph

The following table gives the sum of degrees of adjacent vertices of a given vertex of the 4-semiregular graph given in Fig 5

Table 1. Adjacent vertices

Degree of vertices	Vertices	Adjacent vertices	Sum of degrees of adjacent vertices
deg 1	V7	V4	5
	V8	V6	5
	V9	V1	5
	V10	V3	5
deg 2	V2	V1, V4	10
	V5	V3, V6	10
deg 5	V1	V9, V2, V4, V6, V3	18
	V3	V10, V5, V6, V4, V1	18
	V4	V7, V2, V1, V3, V6	18

3. Connections between Regularity and Sem-iregularity

The connection between regularity and semi-regularity is obtained using the following theorems.

3.1 Theorem

If G is an n -semi regular graph, let G^* be defined as the graph with the same vertex set as G , such that v_1 and v_2 are connected in G^* if and only if they are distance 2 away from each other in G . Then G^* is n -regular.

Proof.

Let G be an n -semi regular graph. Let v be a vertex in G . v is then distance 2 away from exactly other vertices in G . Now consider v in G^* . In G^* , v is connected to exactly those vertices that it was distance 2 away from in G . That is, v is connected to exactly n other vertices. Since this is true for all vertices, G^* is n -regular.

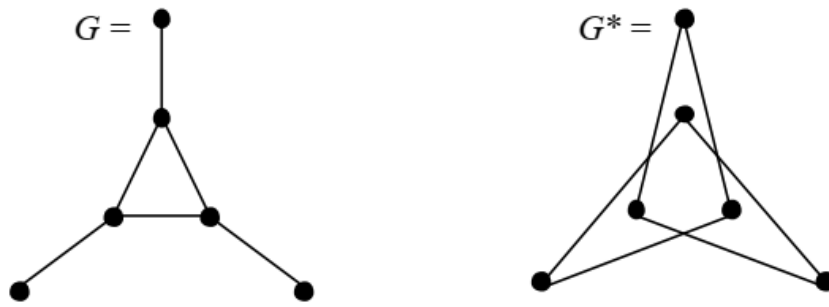


Fig. 6. Shows a graph G and the corresponding G^*

3.2. Theorem

If G is an n -regular graph, let G' is defined by inserting two vertices onto each edge of G . Then G' is an n -semiregular graph.

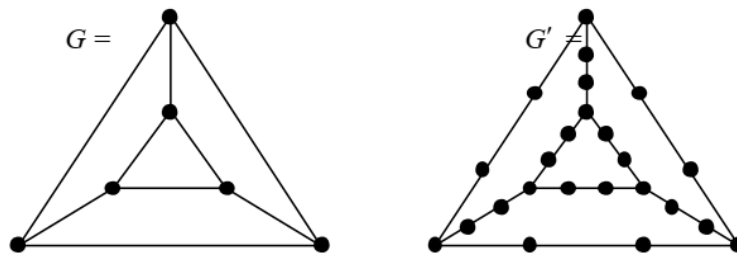
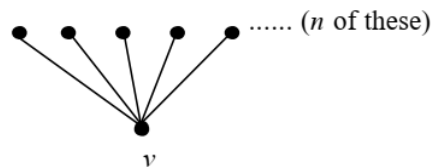


Fig. 7. Shows a graph G and the corresponding G'

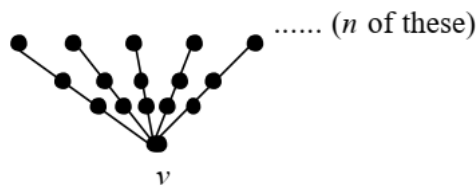
Proof.

Let G be an n -regular graph, and G' as defined above. Let v be a vertex in G' . Then v may or may not have been a vertex in G .

Case 1: If v is a vertex of G , then in G vertex v was connected to exactly n other vertices:

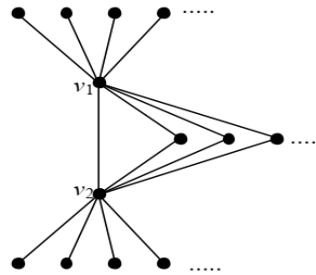


In G' we have:

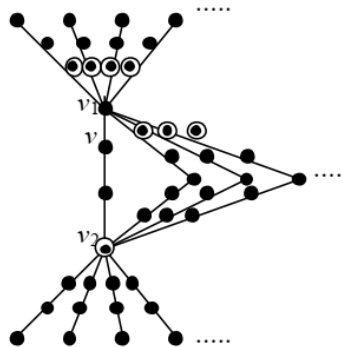


So v is distance 2 away from exactly n other vertices.

Case 2: If v is not a vertex of G , then v must have been added in along an edge of G . Say that v was added to the edge connecting v_1 to v_2 in G . Since G is n -regular, we have the following situation in G :



Thus, in G' we have:



Vertex v is distance two away from exactly n other vertices; those which are highlighted with a double circle above. Since $\deg_2(v)=n$ for every vertex v in G' , G' is n -semi regular.

4. Constructions of Semiregular Graphs

1-semiregular graph must have at least 4 vertices and 3 edges. It can also be noted that if G is 1-semiregular then $\text{dia}(G) \leq 3$. Suppose if $\text{dia}(G) = 4$, then there must be a path from a vertex u to another vertex v of distance 4. The middle vertex of this path will have both u and v at distance 2. Hence if $\text{dia}(G) \geq 4$, then G cannot be 1-semiregular.

This type of restriction is not possible for 2- semi regular graphs. For example, consider the cycle C_n . It can be found that $\text{dia}(C_n) = \lfloor \frac{n}{2} \rfloor$ and C_n is a 2-semiregular graph for all values of n .

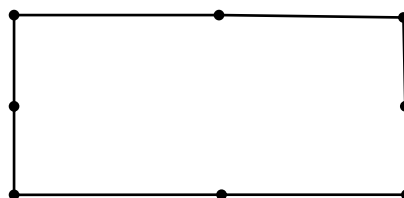


Fig. 8. 2-semiregular graph C_8

4.1 Distance symmetric

A semi regular graph is called Distance Symmetric if the set of vertices can be partitioned in terms of vertices of distance 2.

4.2 Examples

Consider the following semi regular graphs

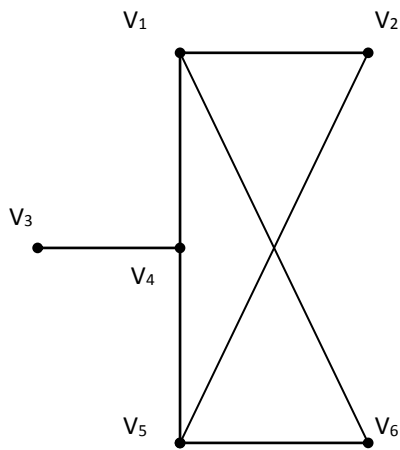


Fig. 9.a 2-semiregular (G_1)

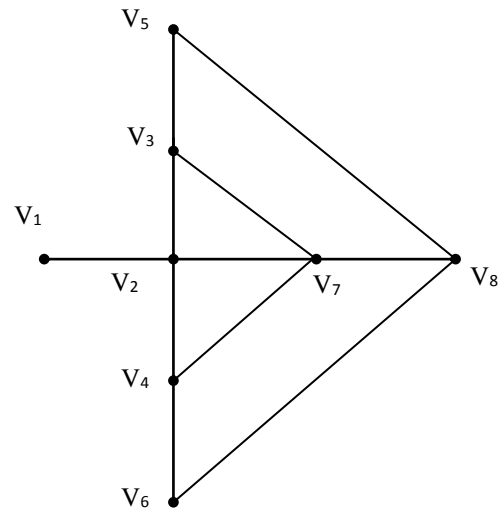


Fig. 9.b 3-semiregular (G_2)

Table 2. Vertex (V) parameters

Vertex(V)	Vertices at distance 2 from V in G_1
V_1	V_3, V_5
V_2	V_4, V_6
V_3	V_1, V_5
V_4	V_2, V_6
V_5	V_1, V_3
V_6	V_2, V_4
Vertex(V)	Vertices at distance 2 from V in G_2
V_1	V_3, V_4, V_7
V_2	V_5, V_6, V_8
V_3	V_1, V_4, V_8
V_4	V_1, V_3, V_8
V_5	V_2, V_6, V_7
V_6	V_2, V_5, V_7
V_7	V_1, V_5, V_6
V_8	V_2, V_3, V_4

From the above table it can be seen that the 2-semiregular graph G_1 is distance symmetric because the vertex set can be partitioned into two sets $A=\{v_1, v_3, v_5\}$; $B = \{ v_2, v_4, v_6 \}$ in terms of vertices of distance 2. But the 3 – semi regular graph G_2 is not distance symmetric as it can be seen that the vertex set cannot be partitioned into subsets of the vertex set of G_2 in terms of distance 2.

4.3 Theorem

The n-Barbell graph is n-semi regular and also distance symmetric.

Proof.

Let G be the n -barbell graph. That is, G is formed by a central line segment connecting u and v with n other vertices connected to each of u and v . Let v be a vertex in G . For $n = 0$, G is 0-semiregular. For all other n , there are two possible cases:

Case 1. v is a point on the central line segment of G . There are $n+1$ vertices connected to v , including u . There are also n other vertices connected to u , and v is distance 2 from each of them. We have considered all the vertices of G , so $\text{deg}_2(v) = n$.

Case 2. v is an endpoint of G . v is distance 2 from u and the other $n-1$ other vertices connected onto v . v is distance 3 from the n vertices connected onto u . We have considered all the vertices of G , so $\text{deg}_2(v) = (n-1)+1 = n$. Thus, $\text{deg}_2(v) = n$ for every vertex v in G , and G is n -semiregular. The proof is in simple way as follows:

Let $V(K_2) = \{u, v\}$; Also let $K_2^{(n)}$ is the graph obtained by adding the pendent vertices w_1, w_2, \dots, w_n at u , and w_{n+1}, \dots, w_{2n} at v . The following table gives the vertices which are at distance 2 from the given vertex.

Table 3. Vertices at distance 2

Vertex	Vertices which are at distance 2
u	w_{n+1}, \dots, w_{2n}
v	w_1, w_2, \dots, w_n
$w_i, i = 1, 2, \dots, n$	$v, w_1, w_2, \dots, w_{i-1}, w_{i+1}, \dots, w_n$
$w_{n+i}, i = 1, 2, \dots, n$	$u, w_{n+1}, \dots, w_{n+i-1}, w_{n+i+1}, \dots, w_{2n}$

Hence every vertex has exactly n vertices at distance 2. Therefore $K_2^{(n)}$ is an n -semiregular graph. Also $V(K_2^{(n)})$ can be partitioned into two sets of vertices $A = \{u, w_{n+1}, \dots, w_{2n}\}$ and $B = \{v, w_1, w_2, \dots, w_n\}$. Thus $K_2^{(n)}$ is distance symmetric.

Note: The n -Barbell graph (n -semi regular graph) so obtained from of K_2 has $2n+2$ vertices and $2n+1$ edges.

4.4 Theorem

An $n-1$ semi regular graph with $2n$ vertices can be constructed from K_n by adding pendent vertices at each vertex of K_n

Proof.

Consider a complete graph K_n , with vertices u_1, u_2, \dots, u_n . Add the pendent vertices v_1, v_2, \dots, v_n respectively at u_1, u_2, \dots, u_n . It can be seen that for every vertex u_i , the $(n-1)$ vertices $v_1, v_2, \dots, v_{(i-1)}, v_{(i+1)}, \dots, v_n$ are at distance 2.

Similarly, Let $d(v_i, v_j) = 1, 1 \leq i \neq j \leq n$. Let $v_{n+1}, v_{n+2}, \dots, v_{2n}$ are the pendent vertices added respectively at v_1, v_2, \dots, v_n . Then for every vertex $v_i, i = 1, 2, \dots, n$, the vertices $v_{n+1}, v_{n+2}, \dots, v_{n+i-1}, v_{n+i+1}, \dots, v_{2n}$ are at distance 2 and also for every other vertex $v_i, i = n+1, \dots, 2n$, the vertices $v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n$ are at distance 2.

Hence the new graph is a $(n-1)$ semi regular.

4.5 Examples

The following graphs are 2-semiregular and 3-semiregular graphs constructed respectively from the complete graphs K_3 and K_4 .

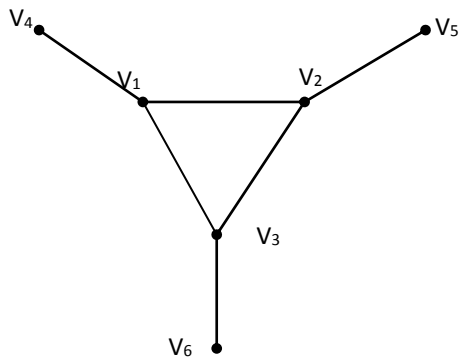


Fig. 10. 2-semiregular from K_3

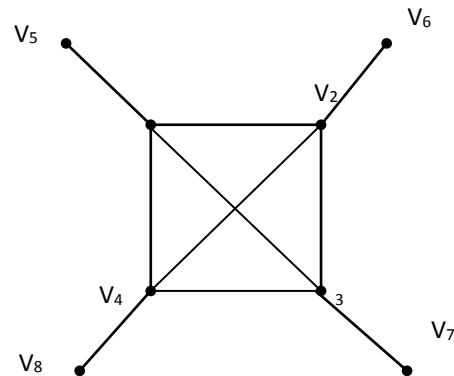


Fig. 11. 3-semiregular from K_4

5. Different Types of Graph Energy

5.1 Adjacency matrix

Let G be a simple graph with n vertices and m edges. Adjacency matrix $n \times n$ of the graph G is given by

$$A(G), (a_{ij}) = \begin{cases} 1 & \text{if } v_i \text{ is adjacency to } v_j \\ 0 & \end{cases}$$

The characteristic polynomial of the above adjacency matrix is given by $P_G(X)$.

The zeros of the polynomial are given by which are eigen values of G .

Here $\lambda_1^{\mu_1}, \lambda_2^{\mu_2}, \dots, \lambda_n^{\mu_n}$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$ and multiplicities $\mu_1, \mu_2, \dots, \mu_n$ are called spectrum of A . The spectrum of A is called spectrum of G .

5.2 Example

consider the graph k_4 .

$$\text{Adjacency matrix is given by } A(k_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Characteristic polynomial $P(k_4, x) = (x + 1)^3(x - 3)$ and the energy of the $E(G) = 6$.

5.3 Energy of G

In the context of spectral graph theory, Energy of a simple graph $G = (V, E)$ with adjacency matrix A is defined as the sum of absolute values of eigen values of A . It is denoted by $E(G)$. More precisely, If G is an

n -vertex graph, then the energy of G is, $E(G) = \sum_{i=1}^n |\lambda_i|$ where λ_i is an eigen values of A , $i = 1, 2, \dots, n$

The total π -electron energy has an expression similar to $E(G)$. Suppose k -eigen values are positive then $E(G) = 2$.

5.4 Bounds for energy of the graph

If G is graph with n -vertices, m -edges and adjacency matrix A then

$$\sqrt{2m + (n-1)n(\det A)^{2/n}} \leq E(G) \leq \sqrt{2mn}.$$

Proof

By Cauchy Schwartz inequality $\left(\sum_{i=1}^n |\lambda_i|\right)^2 \leq n \sum_{i=1}^n |\lambda_i|^2 = 2mn$

5.5 Maximal energy graphs

The classical laplacian matrix of a graph G on n -vertices is atmost $n(1 + \sqrt{n})/2$. Equality holds if and only if G is strongly regular graph with $(n, n(1 + \sqrt{n})/2, (n + 2\sqrt{n})/4)$

5.6 Laplacian matrix

The classical laplacian matrix of a graph G on n -vertices is defined as

$$L(G), (a_{ij}) = \begin{cases} -1 & \text{if } i \neq j, v_i \text{ is adjacency to } v_j \\ 0 & \text{if } i \neq j, v_i \text{ is adjacency not to } v_j \\ d_{ij} & \text{if } i = j \end{cases}$$

In other words, $L(G) = D(G) - A(G)$ where $D(G) = \text{diag}(\text{deg}(v_1), \dots, \text{deg}(v_n))$ and $A(G)$ is the adjacency matrix.

Laplacian spectrum version of graph energy = $\sum_{i=1}^n |\mu_i|$

5.7 Laplacian energy

Let μ_1, \dots, μ_n be the eigen values of $L(G)$, then the laplacian energy $LE(G)$,

$$LE(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|.$$

5.8 Normal laplacian energy

Let μ_1, \dots, μ_n be the eigen values of the normalized Laplacian matrix $L(G)$. The normalized Laplacian energy $NLE(G)$.

We know that, $NLE(G) = \sum_{i=1}^n |\mu_i - 1|$

5.9 Laplacian–energy like

Laplacian spectrum based energy called Laplacian–energy like invariant (LEL) is defined as

$$LEL(G) = \sum_{i=1}^n \sqrt{|\mu_i|}$$

5.10 Sign less laplacian energy

The sign less Laplacian matrix $L^+ = L^+(G)$ is defined as

$$L^+(G), (a_{ij}) = \begin{cases} 1 & \text{if } i \neq j, v_i \text{ is adjacency to } v_j \\ 0 & \text{if } i \neq j, v_i \text{ is adjacency not to } v_j \\ d_{ij} & \text{if } i = j \end{cases}$$

Let μ_1, \dots, μ_n be the eigen values of $L^+(G)$, then the laplacian energy $LE^+(G)$,

$$LE^+(G) = \sum_{i=1}^n \left| \mu_i^+ - \frac{2m}{n} \right|.$$

Also in this case, for regular graphs, $LE^+(G) = E(G)$.

5.11 Q-Laplacian energy

Q- Laplacian matrix of graph G denoted by QE(G). The Q – Laplacian matrix of G(n,m) defined by $Q(G) = D(G) + A(G)$ is the sum of the diagonal matrix of vertex degrees and the adjacency matrix. Let $q_1 \geq q_2 \geq q_3 \geq \dots \geq q_n$ be the Q – Laplacian spectrum of G. Then we define the Q-Laplacian energy of G as

$$QE(G) = \sum_{i=1}^n \left| q_i - \frac{2m}{n} \right|$$

5.12 Distance energy

Let G be a connected graph on n vertices are v_1, v_2, \dots, v_n . The distance matrix of G is the square matrix of order n whose (i,j) th entry is the distance (length of the shortest path) between the vertices v_i and v_j . Let $\rho_1, \rho_2, \dots, \rho_n$ be the eigen values of the distance matrix of the graph G. Then we define $DE = DE(G) =$

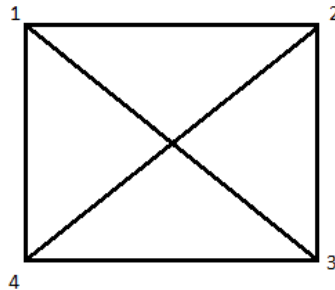
$$\sum_{i=1}^n |\rho_i|.$$

6. Finding Different Types of Energy for Semiregular Graphs

6.1 Example

Let G be a complete graph on four vertices with the edge set $(1,2) (2,3) (2,4) (1,3) (3,4) (1,4)$. So that the graph is 0-semiregular (by theorem)

K_4 :



Adjacency matrix is given by $A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

and $D(G) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Corresponding Laplacian matrix is $L(G) = D(G) - A(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$

To find eigen values of adjacency matrix, $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix}$

Characteristic Polynomial of the adjacent matrix = $(\lambda+1)^3(\lambda-3)$

Eigen values of adjacency matrix are -1, -1, -1, 3 and energy of the graph $E(G) = 6$.

Characteristic Polynomial of the laplacian matrix $|L - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 & -1 \\ -1 & 3-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & -1 \\ -1 & -1 & -1 & 3-\lambda \end{vmatrix} = (\lambda-4)^3(\lambda)$

Eigen values of the laplacian matrix are 4,0,4,4,4

$$LE(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right| = 3|4-3| + |0-3| = 3+3 = 6.$$

$$NLE(G) = \sum_{i=1}^n |\mu_i - 1| = 8$$

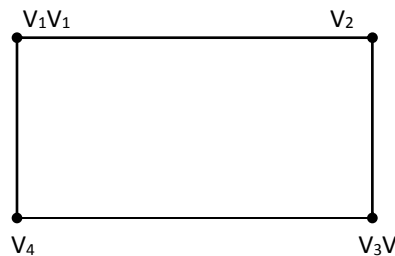
Distance matrix of the graph = $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Table 4. Various energies for graph

Name of the energy	Characteristic polynomial	Eigen values	Energy of graph
Adjacency energy	$(\lambda + 1)^3(\lambda - 3)$	3,-1,-1,-1	6
Laplacian spectrum energy	$\lambda^4 - 12\lambda^3 + 48\lambda^2 - 64\lambda$	4,0,4,4	12
Laplacian energy	$\lambda^4 - 12\lambda^3 + 48\lambda^2 - 64\lambda$	4,0,4,4	6
Normal Laplacian energy	$\lambda^4 - 12\lambda^3 + 48\lambda^2 - 64\lambda$	4,0,4,4	8
Laplacian-energy like	$\lambda^4 - 12\lambda^3 + 48\lambda^2 - 64\lambda$	4,0,4,4	6
Signless energy	$\lambda^4 - 12\lambda^3 + 48\lambda^2 - 80\lambda + 48$	6,2,2,2	12
Distance energy	$\lambda^4 - 6\lambda^2 - 8\lambda - 3$	3,-1,-1,-1	6
Q- Laplacian energy	$\lambda^4 - 12\lambda^3 + 48\lambda^2 - 80\lambda + 48$	6,2,2,2	12

6.2 Example

Let G be a **1-semiregular** cycle graph on four vertices. C_4 is the only 1-semiregular cycle graph.



Adjacency matrix is given by $A(G) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

and
$$D(G) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Corresponding Laplacian matrix is
$$L(G) = D(G) - A(G) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$L(G) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Eigen values C_4 :

The eigen values of $AG) = 0, 0, 2, -2$ and $E(G) = 0$

The eigen values of $L(G)$ are $1 \pm \sqrt{3}, 2 \pm \sqrt{2}$ and the laplacian energy of the G is $LE(G) = 6$.

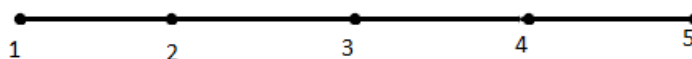
Table 5. various energies for above graph

Name of the energy	Characteristic polynomial	Eigen values	Energy of the graph
Adjacency energy	$\lambda^4 - 4\lambda^2$	0,0,2,-2	0
Laplacian spectrum energy	$\lambda^4 - 6\lambda^3 + 8\lambda^2 + 4\lambda - 4$	$1 \pm \sqrt{3}, 2 \pm \sqrt{2}$	6
Laplacian energy	$\lambda^4 - 6\lambda^3 + 8\lambda^2 + 4\lambda - 4$	$1 \pm \sqrt{3}, 2 \pm \sqrt{2}$	6.2925
Normal Laplacian energy	$\lambda^4 - 6\lambda^3 + 8\lambda^2 + 4\lambda - 4$	$1 \pm \sqrt{3}, 2 \pm \sqrt{2}$	6.2925
Laplacian-energy like	$\lambda^4 - 6\lambda^3 + 8\lambda^2 + 4\lambda - 4$	$1 \pm \sqrt{3}, 2 \pm \sqrt{2}$	5.1216
Signless energy	$\lambda^4 - 6\lambda^3 + 8\lambda^2 + 4\lambda - 4$	$1 \pm \sqrt{3}, 2 \pm \sqrt{2}$	6.2925
Q- Laplacian energy	$\lambda^4 - 6\lambda^3 + 8\lambda^2 + 4\lambda - 4$	$1 \pm \sqrt{3}, 2 \pm \sqrt{2}$	6.2925

6.3 Example

Let G be a directed path on four vertices with the edge set (1,2) (2,3) (3,4) (4,5) which is a 1-semiregular graph from above theorem.

P_5 :



Adjacency matrix is given by $A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

and $D(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$L(G) = D(G) - A(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Laplacian matrix is $L(G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

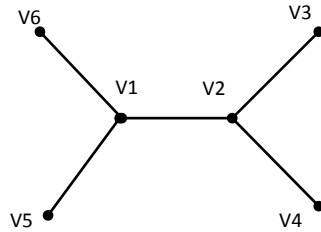
Table 6. various energies for above graph

Name of the energy	Characteristic polynomial	Eigen values	Energy of the graph
Adjacency energy	$-\lambda^5+4\lambda^3-3\lambda$	$0,1,-1,\sqrt{3},-\sqrt{3}$	0
Laplacian spectrum energy	$-\lambda^5+9\lambda^4-28\lambda^3+35\lambda^2-15\lambda+1$	0.081, 0.690, 1.715, 2.831, 3.683	9
Laplacian energy	$-\lambda^5+9\lambda^4-28\lambda^3+35\lambda^2-15\lambda+1$	0.081, 0.690, 1.715, 2.831, 3.683	5.858
Normal Laplacian energy	$-\lambda^5+9\lambda^4-28\lambda^3+35\lambda^2-15\lambda+1$	0.081, 0.690, 1.715, 2.831, 3.683	6.458
Q- Laplacian energy	$-\lambda^5+9\lambda^4-28\lambda^3+35\lambda^2-15\lambda+1$	0.081, 0.690, 1.715, 2.831, 3.683	5.858
Laplacian–energy like	$-\lambda^5+9\lambda^4-28\lambda^3+35\lambda^2-15\lambda+1$	0.081, 0.690, 1.715, 2.831, 3.683	6.0265
Signless energy	$-\lambda^5+9\lambda^4-28\lambda^3+35\lambda^2-15\lambda+1$	0.081, 0.690, 1.715, 2.831, 3.683	5.858

6.4 Example

Let G be an 2-semiregular on six vertices with edge set $(1,2) (2,3) (2,4) (1,5) (1,6)$ so that the complement has 6-cycle from the above theorem. Finding energy of the below graph.

2- semiregular graph:



Adjacency matrix is given by $A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

and $D(G) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Laplacian matrix is $L(G) = D(G) - A(G) = \begin{bmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Q- Laplacian matrix is $Q(G) = D(G) + A(G) = \begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 1 \\ 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Characteristic Polynomial of the adjacent matrix is obtained from

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 & 1 & 1 \\ 1 & -\lambda & 1 & 1 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & -\lambda & 0 & 0 \\ 1 & 0 & 0 & 0 & -\lambda & 0 \\ 1 & 0 & 0 & 0 & 0 & -\lambda \end{vmatrix}$$

The eigen values of $A(G)$ are 0, 0, 1, -1, 2, -2

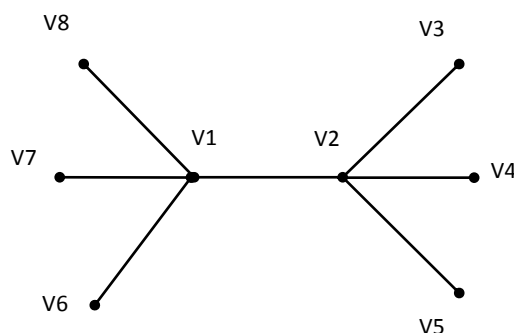
Similarly the eigen values of $L(G)$ 0,1,1,3, $(\pm\sqrt{17} + 5)/2$

Table 7. Various energies for above graph

Name of the energy	Characteristic polynomial	Eigen values	Energy of graph
Adjacency energy	$\lambda^6 - 5\lambda^4 + 4\lambda^2$	0, 0, 1, -1, 2, -2	0
Laplacian spectrum energy	$\lambda^6 - 10\lambda^5 + 34\lambda^4 - 48\lambda^3 + 29\lambda^2 - 6\lambda$	$0, 1, 1, 3, (\pm\sqrt{17} + 5)/2$	10
Laplacian energy	$\lambda^6 - 10\lambda^5 + 34\lambda^4 - 48\lambda^3 + 29\lambda^2 - 6\lambda$	$0, 1, 1, 3, (\pm\sqrt{17} + 5)/2$	8.4564
Normal laplacian energy	$\lambda^6 - 10\lambda^5 + 34\lambda^4 - 48\lambda^3 + 29\lambda^2 - 6\lambda$	$0, 1, 1, 3, (\pm\sqrt{17} + 5)/2$	7.1231
Q- Laplacian energy	$\lambda^6 - 10\lambda^5 + 34\lambda^4 - 48\lambda^3 + 29\lambda^2 - 6\lambda$	$0, 1, 1, 3, (\pm\sqrt{17} + 5)/2$	8.4564
Laplacian-energy like	$\lambda^6 - 10\lambda^5 + 34\lambda^4 - 48\lambda^3 + 29\lambda^2 - 6\lambda$	$0, 1, 1, 3, (\pm\sqrt{17} + 5)/2$	6.5299
Signless energy	$\lambda^6 - 10\lambda^5 + 34\lambda^4 - 48\lambda^3 + 29\lambda^2 - 6\lambda$	$0, 1, 1, 3, (\pm\sqrt{17} + 5)/2$	8.4564

6.5 Example

Let G be a n-Barbell graph which is n-semiregular on eight vertices with edge set (1,2) (2,3) (2,4) (1,5) (1,6) 3-Barbell graph (3- semiregular graph):



Adjacency matrix is given by $A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

and $D(G) = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Laplacian matrix is obtained as

$L(G) = D(G) - A(G) = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Q- Laplacian energy is given by

$Q(G) = D(G) + A(G) = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Characteristic Polynomial of the adjacent matrix is obtained from

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\lambda & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda \end{vmatrix}$$

Table 8. Various energies for above graph

Name of the energy	Characteristic polynomial	Eigen values	Energy of the graph
Adjacency energy	$\lambda^8 - 7\lambda^6 + 9\lambda^4$	0,0,0,0, $\frac{\pm\sqrt{13} \pm 1}{2}$	0
Laplacian spectrum energy	$\lambda^8 - 14\lambda^7 + 72\lambda^6 - 176\lambda^5 + 229\lambda^4 - 162\lambda^3 + 58\lambda^2 - 8\lambda$	0,1,1,1,1,4, $\pm\sqrt{7} + 3$	14
Laplacian energy	$\lambda^8 - 14\lambda^7 + 72\lambda^6 - 176\lambda^5 + 229\lambda^4 - 162\lambda^3 + 58\lambda^2 - 8\lambda$	0,1,1,1,1,4, $\pm\sqrt{7} + 3$	12.2915
Normal laplacian energy	$\lambda^8 - 14\lambda^7 + 72\lambda^6 - 176\lambda^5 + 229\lambda^4 - 162\lambda^3 + 58\lambda^2 - 8\lambda$	0,1,1,1,1,4, $\pm\sqrt{7} + 3$	9.2915
Laplacian-energy like	$\lambda^8 - 14\lambda^7 + 72\lambda^6 - 176\lambda^5 + 229\lambda^4 - 162\lambda^3 + 58\lambda^2 - 8\lambda$	0,1,1,1,1,4, $\pm\sqrt{7} + 3$	8.9712
Signless energy	$\lambda^8 - 14\lambda^7 + 72\lambda^6 - 176\lambda^5 + 229\lambda^4 - 162\lambda^3 + 58\lambda^2 - 8\lambda$	0,1,1,1,1,4, $\pm\sqrt{7} + 3$	12.2915
Q- Laplacian energy	$\lambda^8 - 14\lambda^7 + 72\lambda^6 - 176\lambda^5 + 229\lambda^4 - 162\lambda^3 + 58\lambda^2 - 8\lambda$	0,1,1,1,1,4, $\pm\sqrt{7} + 3$	12.2915

6. Conclusion

In this paper we have studied some of the properties of semi regular. Graph energy has so many application in the field of chemistry, physics and mathematics also. Some types of graph energies are studied for some semi regular graphs in this paper.

Discuss the possible future directions with regards to the following long-standing open problems:

- (i) Existence of semiregular automorphisms,
- (ii) Strongly semiregular and
- (iii) Existence of strongly semi-regular circulant and their relation.

In graph K_4 Laplacian energy different from sign less and Q-Laplacian energy but in other graphs given in the examples has equal values of energy for laplacian energy and sign less energy and Q-Laplacian energy are presented.

Competing Interests

Authors have declared that no competing interests exist.

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