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*Research Article*

# The Integrability of a New Fractional Soliton Hierarchy and Its Application

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Two fractional soliton equations are presented generated from the same spectral problem involved in a fractional potential by the zero-curvature representations. They are a kind of special reductions of the famous AKNS system. The two equations are integrable for they both possess explicit soliton solutions constructed by the $N-$fold Darboux transformation. As an application of the obtained solutions, new soliton solutions of the classic $(2+1)$-dimensional Kadometsev-Petviashvili (KP) equation are soughed out by a cubic polynomial relation. Dynamic properties are analyzed in detail.

## 1. Introduction

Soliton equations occupy an important place in the field of nonlinear science. An obvious character of this kind of equations is that they all have exact solutions. As we known, it is not an easy job to research exact solutions for nonlinear partial differential equations. However, for soliton equations, many methods have been found to study their solutions. The first is the inverse scattering transformation (IST) [1, 2] and the Riemann-Hilbert method [3, 4] which is the modification of the IST. Years of research have shown that the IST has closely relations with other methods such as the algebra-geometric method [5], the Hirota direct technique [6, 7], and the nonlinearization of Lax pairs [8, 9]. In recent years, it is found that many methods actually can be crossed with each other such as using the IST to obtain generalized matrix exponential solutions [10, 11] and using the Hirota technique to solve the nonlinearization systems of Lax pairs [12, 13]. It is worth noting that there have been a lot progress in the IST such as multicomponent IST [14], the classification of solutions [15], and long-time asymptotics of soliton equations with nonzero boundary conditions [16].

Among those approaches, the Darboux transformation (DT) [17–27] turns out to be an efficient method to find explicit solutions, particularly soliton solutions of nonlinear partial systems. The soliton solutions usually are used to describe the particle behavior of solitary waves when they interact. To our surprise, even by a trivial seed complicated solutions of some nonlinear differential equations can be obtained [17–19].

In this paper, by means of the DT, we obtain exact solutions of a fractional soliton hierarchy which actually is a special reduction of the generalized D-AKNS hierarchy [28, 29]. As an application of the resulting solutions, the new solutions to the classic $(2+1)$-dimensional KP equation [30, 31] are combined through a cubic polynomial relation.

The paper is organized as follows. In the next section, beginning with a spectral problem with a fractional potential, two coupled fractional equations are presented as well as the two auxiliary problems. In Section 3, the DT for the resulting fractional soliton equations are constructed. In Section 4, based on a nonzero seed solution, explicit solutions of the fractional soliton equations are obtained. Furthermore, via a cubic polynomial of the obtained solutions, explicit solutions are given for the classic high-dimensional KP equation. Dynamic characters of these solutions are shown in detail. We make some conclusion and discussion of the paper in Section 5.

## 2. A Fractional Soliton Equation Hierarchy

First, let us introduce a fractional soliton equation hierarchy from a $2 \times 2$ spectral problem with a fractional potential.

Consider the following spectral problem

$$\Phi_x = U\Phi, \tag{1}$$

with $\Phi = (\phi_1, \phi_2)^T$ is a column vector and

$$U = \begin{pmatrix} -\lambda + u & 2v \\ 2\left(v^2 + \dfrac{\delta}{v}\right) & \lambda - u \end{pmatrix}, \tag{2}$$

where $\delta$ and $\lambda$ are nonzero constant and spectral parameter, respectively, and $u$, $v$ as the potentials are smooth functions owning variables $x, y, t$. It is a special reduction of D-AKNS spectral problem [28, 29]

$$\begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda + u & p \\ q & \lambda - u \end{pmatrix} \begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{pmatrix}, \tag{3}$$

with the relation $p = 2v, q = 2(v^2 + (\delta/v))$. The Hamiltonian structure of the D-AKNS equation was deduced through the gauge transformation [28] and the trace identity [29], respectively.

Actually, in [25, 32], a fractional spectral problem called generalized coupled KdV one has been presented already

$$\begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda + u & 2v \\ 2\left(l + \dfrac{r}{v}\right) & \lambda - u \end{pmatrix} \begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{pmatrix}, \tag{4}$$

where $l$ and $r$ are constants. Our spectral problem and the corresponding DT are obviously different with those in [25].

From the spectral problem (1), we could construct isospectral and nonisospectral soliton equation hierarchies which form the $\tau$-symmetry algebra [33, 34] depending on the relations between the spectral parameter $\lambda$ and the time variable $t$. Here, we only give the first two isospectral equations of the hierarchy. Considering the two auxiliary problems of the spectral problem (1)

$$\Phi_y = V_1\Phi, \tag{5a}$$

$$\Phi_t = V_2\Phi, \tag{5b}$$

with

$$V_1 = \begin{pmatrix} -\lambda^2 + V_{10}^{(11)} & 2v\lambda + V_{10}^{(12)} \\ 2\left(v^2 + \dfrac{\delta}{v}\right)\lambda + V_{10}^{(21)} & \lambda^2 - V_{10}^{(11)} \end{pmatrix},$$

$$V_2 = \begin{pmatrix} -\lambda^3 + 2(v^3 + \delta)\lambda + V_{20}^{(11)} & 2v\lambda^2 + (2uv - v_x)\lambda + V_{20}^{(12)} \\ 2\left(v^2 + \dfrac{\delta}{v}\right)\lambda^2 + V_{21}^{(21)}\lambda + V_{20}^{(21)} & \lambda^3 - 2(v^3 + \delta)\lambda - V_{20}^{(11)} \end{pmatrix}, \tag{6}$$

where

$$\begin{aligned}
V_{10}^{(12)} &= 2uv - v_x, \\
V_{10}^{(21)} &= 2u\left(v^2 + \frac{\delta}{v}\right) + 2vv_x - \frac{\delta v_x}{v^2}, \\
V_{10}^{(11)} &= u^2 - \frac{u_x}{6v^3}(v^3 - 2\delta) + \frac{v_x^2}{6v^5}(\delta + v^3) - \frac{v_{xx}}{6v^4}(\delta - 2v^3), \\
V_{21}^{(21)} &= \frac{2\delta u}{v} + 2uv^2 - \frac{\delta v_x}{v^2} + 2vv_x, \\
V_{20}^{(12)} &= 2u^2v - 4\delta v - 4v^4 - vu_x - 2uv_x + \frac{v_{xx}}{2}, \\
V_{20}^{(11)} &= u^3 - 2\delta u - 2uv^3 - \frac{uu_x}{2} + \frac{\delta uu_x}{v^3} - \frac{2\delta v_x}{v} + v^2v_x - \frac{\delta u_xv_x}{2v^4} + \frac{u_xv_x}{v} + \frac{\delta uv_x^2}{2v^5} + \frac{uv_x^2}{2v^2} - \frac{\delta v_x^3}{4v^6} + \frac{u_{xx}}{4} - \frac{\delta uv_{xx}}{2v^4} + \frac{uv_{xx}}{v} + \frac{\delta v_xv_{xx}}{4v^5} + \frac{v_xv_{xx}}{4v^2}, \\
V_{20}^{(21)} &= \frac{2\delta u^2}{v} - \frac{4\delta^2}{v} - 8\delta v^2 + 2u^2v^2 - 4v^5 + \frac{\delta u_x}{v} + v^2u_x - \frac{2\delta uv_x}{v^2} + 4uvv_x + v_x^2 - \frac{\delta v_{xx}}{2v^2} + \frac{\delta v_x^2}{v^3} + vv_{xx}.
\end{aligned} \tag{7}$$

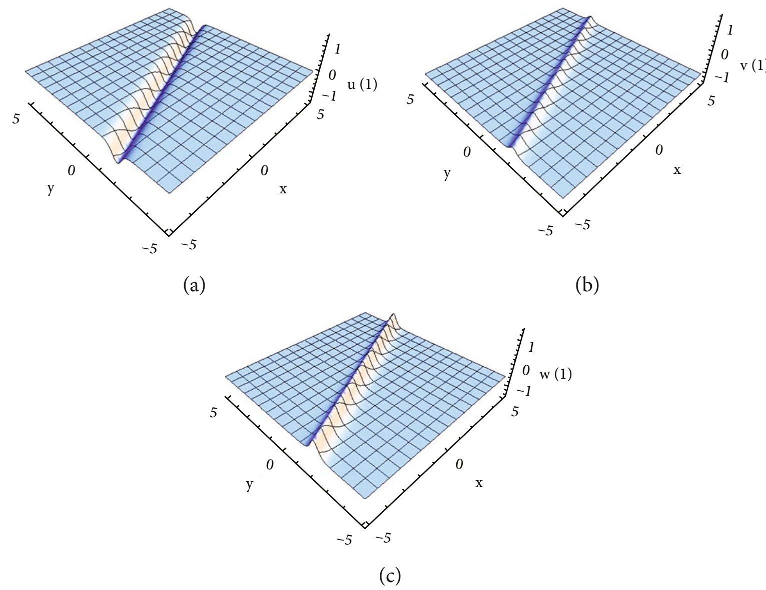


Figure 1: One soliton solutions with $\hat{u} = 0$, $\hat{v} = -1$, $\lambda_1 = -2.1$, $\gamma_1 = -0.5$, $\delta = 0.5$, and $t = 0$.

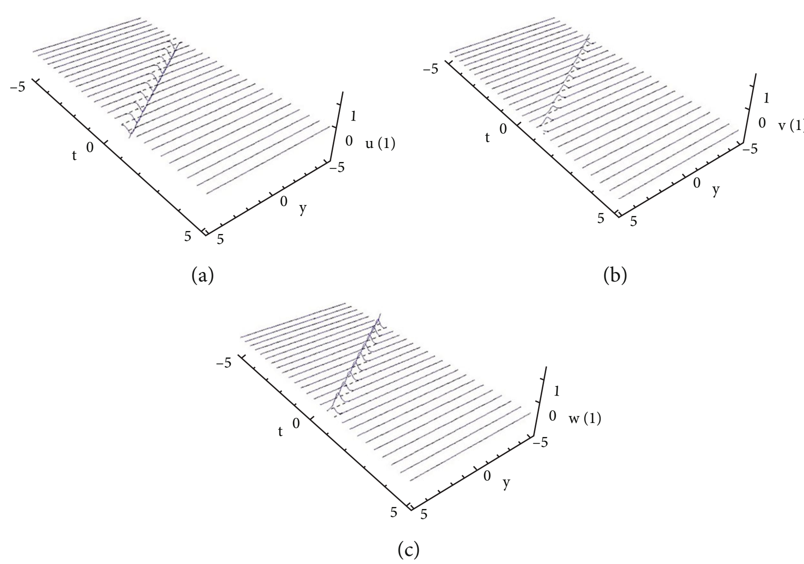


Figure 2: Spreadings of one soliton solutions with $\hat{u} = 0$, $\hat{v} = -1$, $\lambda_1 = 2.1$, $\gamma_1 = -0.5$, $\delta = 0.5$, and $x = 3$.

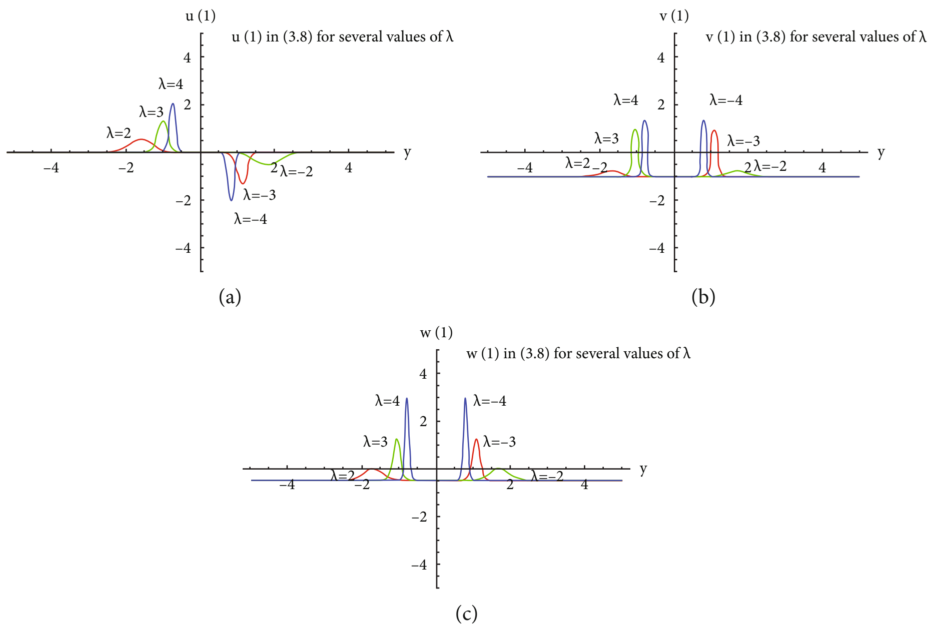


FIGURE 3: One-soliton solutions for different values of $\lambda$ with $\widehat{u}=0$, $\widehat{v}=-1, \gamma_1=-0.5$, $\delta=0.5$, $x=3$, and $t=0$.

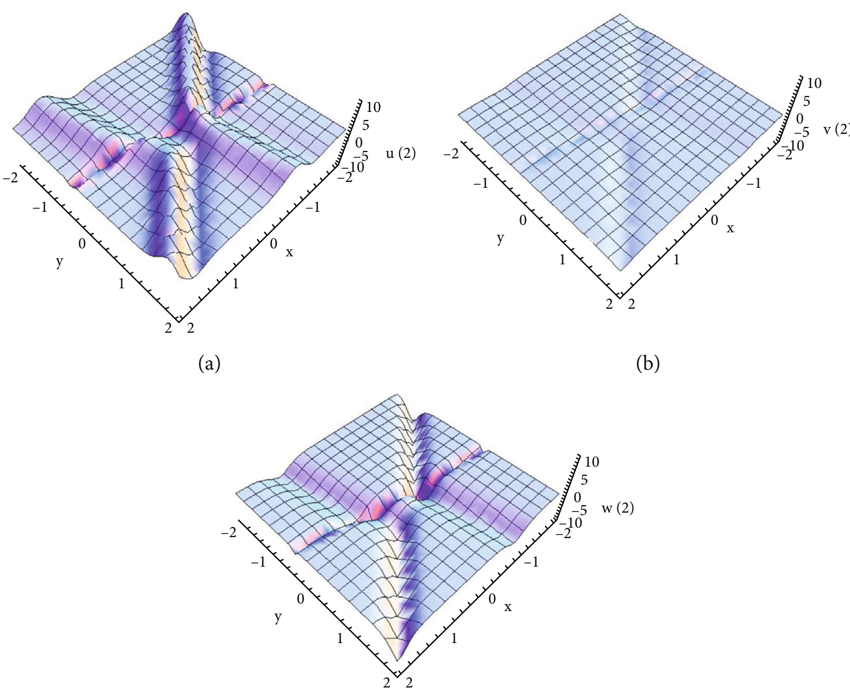


FIGURE 4: Interactions of three-soliton solutions at $t=0$ with $\widehat{u}=2$, $\widehat{v}=-1$, $\lambda_1=-3$, $\lambda_2=5$, $\lambda_3=-1.8$, $\gamma_1=-1.2$, $\gamma_2=-0.15$, $\gamma_3=0.5$, and $\delta=0.5$.

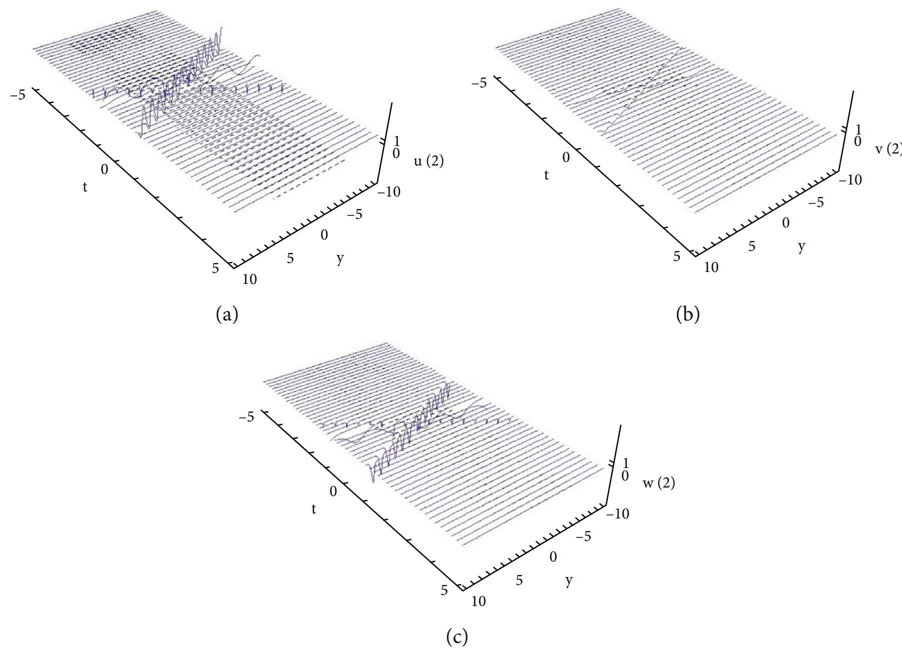


Figure 5: Spreadings of three-soliton solutions with $\hat{u} = 2$, $\hat{v} = -1$, $\lambda_1 = -3$, $\lambda_2 = 5$, $\lambda_3 = -1.8$, $\gamma_1 = -1.2$, $\gamma_2 = -0.15$, $\gamma_3 = 0.5$, $\delta = 0.5$, and $x = 1$.

The zero-curvature equations

$$\begin{aligned} U_y - V_{1x} + [U, V_1] &= 0 \\ U_t - V_{2x} + [U, V_2] &= 0 \end{aligned} \tag{8}$$

give the first two fractional soliton equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} \dfrac{1}{6}\partial_x \left[\dfrac{1}{v^2} g_3^1 + \left(\dfrac{2}{v} - \dfrac{\delta}{v^4}\right) g_3^2\right] \\ \dfrac{1}{3v} g_3^1 - \dfrac{1}{3}\left(1 + \dfrac{\delta}{v^3}\right) g_3^2 \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} \dfrac{1}{6}\partial_x \left[\dfrac{1}{v^2} g_4^1 + \left(\dfrac{2}{v} - \dfrac{\delta}{v^4}\right) g_4^2\right] \\ \dfrac{1}{3v} g_4^1 - \dfrac{1}{3}\left(1 + \dfrac{\delta}{v^3}\right) g_4^2 \end{pmatrix}, \tag{10}$$

where

$$
\begin{aligned}
g_3^1 &= v_x^2\left(\frac{\delta}{v^3}+1\right)-\left(\frac{\delta}{v^2}-2v\right)\left(2uv_x+\frac{v_{xx}}{2}\right)+\left(\frac{\delta}{v}+v^2\right)\left(u_x+2u^2-4\delta\right)-4v^2\left(\delta+v^3\right),\\
g_3^2 &= \frac{v_{xx}}{2}-4\delta v+2u^2v-4v^4-vu_x-2uv_x,\\
g_4^1 &= \left(\frac{\delta}{v}+v^2\right)\left[\frac{3v_x}{v^2}\left(uv_x+\frac{v_{xx}}{2}\right)+2u^3-12\delta u+3uu_x+\frac{u_{xx}}{2}\right]-12v^2\left(\delta u+uv^3+v^2v_x\right)+6\delta v_x\left(\frac{\delta}{v^2}-v\right)-\left(\frac{\delta}{v^2}-2v\right)\left(3u^2v_x+\frac{3u_xv_x}{2}+\frac{v_{xxx}}{4}+\frac{3uv_{xx}}{2}\right)-\frac{3\delta v_x^3}{2v^4},\\
g_4^2 &= 12uv\left(\frac{u^2}{6}-\delta-v^3-\frac{u_x}{4}\right)+6v_x\left(\delta-\frac{u^2}{2}+v^3+\frac{u_x}{4}\right)+\frac{vu_{xx}}{2}+\frac{3uv_{xx}}{2}-\frac{v_{xxx}}{4},
\end{aligned}
\tag{11}
$$

and $\partial_x^{-1}$ defined by $\partial_x^{-1}f(x,y,t)=\int_{-\infty}^{x}f(s,y,t)ds$ is the inverse operator of $\partial$ under the decaying condition at infinity.

Notice that (9) and (10) are two fractional soliton equations. It is very interesting that the $N$-soliton solutions of the two equations can be acquired by the DT. So they are integrable. Need to point out that our equations belong to classic soliton equations rather than fractional order derivative soliton equations.

## 3. The DT for the Two Fractional Soliton Equations

Now, let us use the method in [35–37] to construct the gauge transformations of the spectral problem (1). Assume a gauge transformation of (1) is

$$\bar{\Phi}=T\Phi. \tag{12}$$

Then, the new spectral function $\bar{\Phi}$ should satisfy

$$\bar{\Phi}_x=\bar{U}\Phi, \tag{13}$$

$$\bar{\Phi}_y=\bar{V}_1\Phi, \tag{14}$$

$$\bar{\Phi}_t=\bar{V}_2\Phi, \tag{15}$$

where $\bar{U}$, $\bar{V}_1$ and $\bar{V}_2$ are three matrices with the same form as $U$, $V_1$, $V_2$, respectively. It is easy to find that $T$ is determined by the following three matrix equations:

$$T_x+TU=\bar{U}T, \tag{16}$$

$$T_y+TV_1=\bar{V}_1T, \tag{17}$$

$$T_t+TV_2=\bar{V}_2T. \tag{18}$$

Assume

$$T=T(\lambda)=\alpha\begin{pmatrix}\lambda^N+T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}, \tag{19}$$

where $T_{jl}=\sum_{k=0}^{N-1}T_{jl}^k\lambda^k\,(j,l=1,2)$ and $\alpha$, $T_{jl}^k$ are functions of variables $x$, $y$, and $t$.

Let $\lambda=\lambda_j$ the spectral problem (1) is a matrix differential equation. Since $U$ is a $2\times 2$ matrix, we can suppose that its corresponding two basic vector solutions are

$$
\begin{aligned}
\varphi(\lambda_j)&=\left(\varphi_1(\lambda_j),\varphi_2(\lambda_j)\right)^T,\\
\psi(\lambda_j)&=\left(\psi_1(\lambda_j),\psi_2(\lambda_j)\right)^T,
\end{aligned}
\tag{20}
$$

where $T$ is an abuse notation meaning the transposition of a matrix. From the gauge transformation (12), we know that the basic solutions have following relations

$$
\begin{cases}
\left(\lambda_j^N+T_{11}\right)\varphi_1(\lambda_j)+T_{12}\varphi_2(\lambda_j)-\gamma_j\left[\left(\lambda_j^N+T_{11}\right)\psi_1(\lambda_j)+T_{12}\psi_2(\lambda_j)\right]=0,\\
T_{21}\varphi_1(\lambda_j)+T_{22}\varphi_2(\lambda_j)-\gamma_j\left(T_{21}\psi_1(\lambda_j)+T_{22}\psi_2(\lambda_j)\right)=0,
\end{cases}
\tag{21}
$$

where $\gamma_j(1\le j\le 2N-1)$ is a constant.

Setting

$$\sigma_j=\frac{\varphi_2(\lambda_j)-\gamma_j\psi_2(\lambda_j)}{\varphi_1(\lambda_j)-\gamma_j\psi_1(\lambda_j)},\; j=1,2,\cdots,2N-1, \tag{22}$$

the relation (21) can be translated into the following system of linear algebraic equations

$$
\begin{cases}
\lambda_j^N+T_{11}+\sigma_jT_{12}=0,\\
T_{21}+\sigma_jT_{22}=0,
\end{cases}
\tag{23}
$$

i.e.,

$$
\begin{cases}
\displaystyle\sum_{k=0}^{N-1}\left(T_{11}^k+\sigma_jT_{12}^k\right)\lambda_j^k=-\lambda_j^N,\\
\displaystyle\sum_{k=0}^{N-1}\left(T_{21}^k+\sigma_jT_{22}^k\right)\lambda_j^k=0.
\end{cases}
\tag{24}
$$

In order to make equations (23) and (24) possess nonzero solution, the spectral parameters $\lambda_j$ and constants $\gamma_j$ ($\lambda_k\ne\lambda_j$ when $k\ne j$) should be selected carefully. To this end, taking $T_{12}^{N-1}=-v$, $\beta_{21}^{N-1}=\sqrt[3]{v^3+T_{11,x}^{N-1}/2}$, and

$$-T_{21}^{N-1}=\left(\beta_{21}^{N-1}\right)^2+\frac{\delta}{\beta_{21}^{N-1}}, \tag{25}$$

from (23) and (24), the other element $T_{jl}^k$ $(j,l=1,2,k=0,1,\cdots,N-1)$ can be uniquely identified; furthermore, $\alpha$ can be deduced out too.

Through equation (19), it is easily found that the determinant of $T(\lambda)$ is a $(2N-1)$-th order polynomial of $\lambda$. Especially,

$$\det T(\lambda_j)=\alpha^2\left[\lambda_j^N T_{22}(\lambda_j)+T_{11}(\lambda_j)T_{22}(\lambda_j)-T_{12}(\lambda_j)T_{21}(\lambda_j)\right], \tag{26}$$

where det means the determinant of a square matrix. But from (23) and (24), we have the following relations:

$$\begin{aligned}\lambda_j^N+T_{11}(\lambda_j)&=-\sigma_j T_{12}(\lambda_j),\\ T_{21}(\lambda_j)&=-\sigma_j T_{22}(\lambda_j),\end{aligned} \tag{27}$$

which implies that

$$\det T(\lambda)=\beta\prod_{j=1}^{2N-1}(\lambda-\lambda_j). \tag{28}$$

In other words, $\lambda_j (j=1,2,\cdots,2N-1)$ satisfies the equation $\det T(\lambda)=0$ (where $\beta$ has nothing with $\lambda$). In a way similar to [35–37], we can verify the following propositions.

**Proposition 1.** *Let $\alpha$ and $T_{22}^{N-1}$ satisfy*

$$\alpha^2 T_{22}^{N-1}=const. \tag{29}$$

*the matrix $\bar{U}$ in (16) owns the following form:*

$$\bar{U}=\begin{pmatrix}-\lambda+\bar{u} & 2\bar{v}\\ 2\left(\bar{v}^2+\dfrac{\delta}{\bar{v}}\right) & \lambda-\bar{u}\end{pmatrix}, \tag{30}$$

*which is just the form of $U$. The Bäcklund transformation (BT) between the new potentials $\bar{u}$ and $\bar{v}$ and the old potentials $u$ and $v$ is*

$$\begin{aligned}\bar{u}&=u-\frac{1}{2}\partial_x\ln\left(T_{22}^{N-1}\right),\\ \bar{v}^3&=v^3-\frac{1}{2}T_{11,x}^{N-1}.\end{aligned} \tag{31}$$

*Proof.* Since $T^{-1}\det T=T^*$, assume

$$(T_x+TU)T^*=\begin{pmatrix}f_{11}(\lambda) & f_{12}(\lambda)\\ f_{21}(\lambda) & f_{22}(\lambda)\end{pmatrix}, \tag{32}$$

where $T^*$ and $T^{-1}$ are the adjoint and inverse matrix of $T$, respectively. It is clear that $f_{jl}(\lambda)(j,l=1,2)$ is the polynomial of $\lambda$. Simple computation shows that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are $2N$-th order ones, while $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are $(2N-1)$-th order ones.

On the other hand, because $\det T(\lambda)$ is a $(2N-1)$-th polynomial, we suppose

$$(T_x+TU)T^*=P(\lambda)\det T, \tag{33}$$

where

$$P(\lambda)=\begin{pmatrix}\lambda P_{11}^{(1)}+P_{11}^{(0)} & P_{12}^{(0)}\\ P_{21}^{(0)} & \lambda P_{22}^{(1)}+P_{22}^{(0)}\end{pmatrix}, \tag{34}$$

with $P_{kj}^{(0)}$ and $P_{kk}^{(1)}(k,j\in\{1,2\})$ having no relation with $\lambda$. It is easy to see that $f_{sl}(\lambda)=0$ has a solution $\lambda_j(s,l=1,2)$ by (33). Furthermore, we have

$$T_x+TU=P(\lambda)T \tag{35}$$

by rewriting (33). The coefficient of $\lambda^k(k=N-1,N,N+1)$ in (35) gives

$$\begin{aligned}P_{11}^{(1)}&=-1,\\ P_{22}^{(1)}&=1,\\ P_{21}^{(0)}&=-2T_{21}^{N-1},\\ P_{11}^{(0)}&=u+(\ln\alpha)_x,\\ P_{12}^{(0)}T_{21}^{N-1}&=T_{11,x}^{N-1}-2\left(v^3+\delta\right),\\ P_{22}^{(0)}&=(\ln\alpha)_x+\partial_x\ln\left(T_{22}^{N-1}\right)-u.\end{aligned} \tag{36}$$

From (29), we have $\partial_x\ln\alpha=-(1/2)\partial_x\ln\left(T_{22}^{N-1}\right)$. Submitting it into the above equations yields

$$\begin{aligned}P_{11}^{(0)}&=\bar{u},\\ P_{12}^{(0)}&=2\bar{v},\\ P_{22}^{(0)}&=-\bar{u},\\ P_{21}^{(0)}&=2\left(\bar{v}^2+\frac{\delta}{\bar{v}}\right).\end{aligned} \tag{37}$$

Noticing that $P(\lambda)$ and $U$ have the same form, so we conclude that $\bar{U}=P(\lambda)$. □

Next, let us suppose that $\varphi(\lambda_j)$ and $\psi(\lambda_j)$ are solutions of (5a) at the same time. We will prove that under the transformations (31), $\bar{V}_1$ in (17) has the identical form as $V_1$ through a similar way.

**Proposition 2.** *Suppose that $\alpha$ satisfies a differential equation*

$$\partial_y \ln \alpha = \bar{V}_1^{(11)} - V_1^{(11)} + 2v^3 - 2\bar{v}^3, \tag{38}$$

*where*

$$\bar{V}_1^{(11)} = \bar{u}^2 - \frac{\bar{u}_x}{6} + \frac{\delta \bar{u}_x}{3\bar{v}^3} + \frac{\delta \bar{v}_x^2}{6\bar{v}^5} + \frac{\bar{v}_x^2}{6\bar{v}^2} - \frac{\delta \bar{v}_{xx}}{6\bar{v}^4} + \frac{\bar{v}_{xx}}{3\bar{v}}, \tag{39}$$

*then $\bar{V}_1^{(11)}$ and $\bar{V}_1$ have the same form as $V_1^{(11)}$ and $V_1$ except changing $u, v$ as well as their derivatives into $\bar{u}, \bar{v}$ and their derivatives. That is, the new potentials $\bar{u}$ and $\bar{v}$ are obtained from old potentials $u$ and $v$ according to the same BT (31).*

*Proof.* Same as in Proposition 1, taking

$$\left(T_y + TV_1\right)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \tag{40}$$

the elements of the above matrix are polynomials of $\lambda$ with the diagonals are $(2N+1)$-th order and the off-diagonals are $(2N)$-th order. In consideration of (40), we suppose

$$\left(T_y + TV_1\right)T^* = Q(\lambda)\det T, \tag{41}$$

where

$$Q(\lambda) = \begin{pmatrix} \sum_{l=0}^{2} q_{11}^{(l)}\lambda^l & \sum_{l=0}^{1} q_{12}^{(l)}\lambda^l \\ \sum_{l=0}^{1} q_{21}^{(l)}\lambda^l & \sum_{l=0}^{2} q_{22}^{(l)}\lambda^l \end{pmatrix}, \tag{42}$$

where $q_{kj}^{(l)}(k, j = 1, 2)$ has no relation with $\lambda$. Then, $\lambda_j$ $(j = 1, \cdots, 2N-1)$ is the root of the polynomial $g_{sl}(\lambda)$ in the matrix (41) $(s, l = 1, 2)$. Now, let us rewrite (41) as

$$T_y + TV_1 = Q(\lambda)T. \tag{43}$$

The coefficients of $\lambda^k (k = N-1, N, N+1, N+2)$ in (43) generate

$$\begin{aligned}
q_{11}^{(2)} &= -1, \\
q_{22}^{(2)} &= 1, \\
q_{11}^{(1)} &= q_{22}^{(1)} = 0, \\
q_{21}^{(1)} &= -2T_{21}^{N-1} = 2\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right), \\
q_{11}^{(0)} + \left(q_{12}^{(1)} + 1\right)T_{11}^{N-2} + q_{12}^{(1)}T_{21}^{N-1} &= V_1^{(11)} + 2\left(v^2 + \frac{\delta}{v}\right)T_{12}^{N-1} + \partial_y \ln \alpha, \\
q_{11}^{(1)}T_{12}^{N-1} + \left(q_{11}^{(2)} - 1\right)T_{12}^{N-2} + q_{12}^{(1)}T_{22}^{N-1} &= 2vT_{11}^{N-1} + V_1^{(12)}, \\
q_{21}^{(0)} + q_{21}^{(1)}T_{11}^{N-1} + q_{22}^{(1)}T_{21}^{N-1} + \left(q_{22}^{(2)} + 1\right)T_{21}^{N-2} &= 2\left(v^2 + \frac{\delta}{v}\right)T_{22}^{N-1}, \\
q_{21}^{(1)}T_{12}^{N-1} + q_{22}^{(1)}T_{22}^{N-1} + \left(q_{22}^{(2)} - 1\right)T_{22}^{N-2} &= 2vT_{21}^{N-1}, \\
\left(q_{11}^{(0)} - V_1^{(11)}\right)T_{11}^{N-1} &= V_1^{(21)}T_{12}^{N-1} - q_{11}^{(1)}T_{11}^{N-2} - q_{12}^{(0)}T_{21}^{N-1} - q_{12}^{(1)}T_{21}^{N-2} - \left(q_{11}^{(2)} + 1\right)T_{11}^{N-3} + 2\left(v^2 + \frac{r}{v}\right)T_{12}^{N-2} + \frac{\left(\alpha T_{11}^{N-1}\right)_y}{\alpha}, \\
q_{12}^{(0)}T_{22}^{N-1} &= \left(V_1^{(22)} - q_{11}^{(0)}\right)T_{12}^{N-1} - q_{11}^{(1)}T_{12}^{N-2} - \left(q_{11}^{(2)} - 1\right)T_{12}^{N-3} - q_{12}^{(1)}T_{22}^{N-2} + 2vT_{11}^{N-2} + V_1^{(12)}T_{11}^{N-1} + \frac{\left(\alpha T_{12}^{N-1}\right)_y}{\alpha}, \\
q_{21}^{(0)}T_{11}^{N-1} &= \left(V_1^{(11)} - q_{22}^{(0)}\right)T_{21}^{N-1} - q_{21}^{(1)}T_{11}^{N-2} + q_{22}^{(1)}T_{21}^{N-2} - \left(q_{22}^{(2)} + 1\right)T_{21}^{N-3} + V_1^{(21)}T_{22}^{N-1} + 2\left(v^2 + \frac{r}{v}\right)T_{22}^{N-2} + \frac{\left(\alpha T_{21}^{N-1}\right)_y}{\alpha}, \\
\left(q_{22}^{(0)} - V_1^{(22)}\right)T_{22}^{N-1} &= -q_{21}^{(0)}T_{12}^{N-1} - q_{21}^{(1)}T_{12}^{N-2} - q_{22}^{(1)}T_{22}^{N-2} - \left(q_{22}^{(2)} - 1\right)T_{22}^{N-3} + 2vT_{21}^{N-2} + V_1^{(12)}T_{21}^{N-1} + \frac{\left(\alpha T_{22}^{N-1}\right)_y}{\alpha}.
\end{aligned} \tag{44}$$

Furthermore, to balance the equality (35), we let the coefficients of $\lambda^{N-1}$ and $\lambda^{N-2}$ satisfy

$$\begin{aligned}
T^{N-1}_{11,x} &= 2\bar{v}T^{N-1}_{21} - 2\left(v^2 + \frac{\delta}{v}\right)T^{N-1}_{12},\\
T^{N-1}_{12,x} &= 2uT^{N-1}_{12} - 2T^{N-2}_{12} - 2vT^{N-1}_{11} + 2\bar{v}T^{N-1}_{22},\\
T^{N-1}_{21,x} &= 2T^{N-2}_{21} + 2\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right)T^{N-1}_{11} - 2\left(v^2 + \frac{\delta}{v}\right)T^{N-1}_{22} - 2\bar{u}T^{N-1}_{21},\\
T^{N-1}_{22,x} &= 2\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right)T^{N-1}_{12} - 2vT^{N-1}_{21} + 2uT^{N-1}_{22} - 2\bar{u}T^{N-1}_{22},\\
T^{N-2}_{12,x} &= 2uT^{N-2}_{12} - 2T^{N-3}_{12} - 2vT^{N-2}_{11} + 2\bar{v}T^{N-2}_{22},\\
T^{N-2}_{21,x} &= 2T^{N-3}_{21} - 2\bar{u}T^{N-2}_{21} - 2\left(v^2 + \frac{\delta}{v}\right)T^{N-2}_{22} + 2\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right)T^{N-2}_{11}.
\end{aligned} \tag{45}$$

By (38), (25), and the above equalities, we find that

$$\begin{aligned}
q^{(0)}_{11} &= -q^{(0)}_{22} = \bar{u}^2 - \frac{\bar{u}_x}{6} + \frac{\delta\bar{u}_x}{3\bar{v}^3} + \frac{\delta\bar{v}_x^2}{6\bar{v}^5} + \frac{\bar{v}_x^2}{6\bar{v}^2} - \frac{\delta\bar{v}_{xx}}{6\bar{v}^4} + \frac{\bar{v}_{xx}}{3\bar{v}},\\
q^{(0)}_{12} &= 2\bar{u}\bar{v} - \bar{v}_x,\\
q^{(1)}_{12} &= 2\bar{v},\\
q^{(0)}_{21} &= 2\bar{u}\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right) - \frac{\delta\bar{v}_x}{\bar{v}^2} + 2\bar{v}\bar{v}_x.
\end{aligned} \tag{46}$$

By (17) and (43), one can assert that $Q(\lambda)$ and $V_1$ have the same form, i.e., $\bar{V}_1 = Q(\lambda)$. □

According to Proposition 1 and Proposition 2, the Lax pairs (1) and (7) are transformed into another Lax pairs (13) and (14), respectively. Accordingly, a new zero curvature equation $\bar{U}_y - \bar{V}_{1,x} + [\bar{U}, \bar{V}_1] = 0$ is generated by the compatibility condition $\bar{\phi}_{xy} = \bar{\phi}_{yx}$. Therefore, a couple new solution $\bar{u}, \bar{v}$ to the soliton equation (9) is constructed also. That is, equation (9) can be deduced from both the old Lax pairs and new ones. The transformation $(\phi, u, v) \longrightarrow (\bar{\phi}, \bar{u}, \bar{v})$ is named a DT of the fractional soliton equation (9).

Based on the above analysis, here is our theorem.

**Theorem 3.** *Under the DT (12) and (31) , the new solution $(\bar{u}, \bar{v})$ of (9) is obtained from its old one $(u, v)$, where $T^{N-1}_{11}$ and $T^{N-1}_{22}$ are determined by (25) and the linear algebraic equations (23) and (24).*

Utilizing the similar way of verifying Proposition 1 and Proposition 2, we have a conclusion that $\bar{V}_2$ has the same form as $V_2$ under the transformation (31).

**Proposition 4.** *Let $\alpha$ be a solution of the following ordinary differential equation*

$$\begin{aligned}
\partial_t \ln\alpha = {}& \bar{V}^{(11)}_2 - V^{(11)}_2 - 4\delta\bar{u} - 4\bar{u}\bar{v}^3 + \frac{2\delta\bar{v}_x}{\bar{v}} - \bar{v}^2\bar{v}_x + 4\delta u\\
&+ 4uv^3 - \frac{2\delta v_x}{v} + v^2 v_x,
\end{aligned} \tag{47}$$

*with*

$$\begin{aligned}
\bar{V}^{(11)}_2 = {}& \bar{u}^3 - 2\delta\bar{u} - 2\bar{u}\bar{v}^3 - \frac{1}{2}\bar{u}\bar{u}_x + \frac{\delta\bar{u}\bar{u}_x}{\bar{v}^3} - \frac{2\delta\bar{v}_x}{\bar{v}} + \bar{v}^2\bar{v}_x\\
&- \frac{\delta\bar{u}_x\bar{v}_x}{2\bar{v}^4} + \frac{\bar{u}_x\bar{v}_x}{\bar{v}} + \frac{\delta\bar{u}\bar{v}_x^2}{2\bar{v}^5} + \frac{\bar{u}\bar{v}_x^2}{2\bar{v}^2} - \frac{\delta\bar{v}_x^3}{4\bar{v}^6} + \frac{\bar{u}_{xx}}{4}\\
&- \frac{\delta\bar{u}\bar{v}_{xx}}{2\bar{v}^4} + \frac{\bar{u}\bar{v}_{xx}}{\bar{v}} + \frac{\delta\bar{v}_x\bar{v}_{xx}}{4\bar{v}^5} + \frac{\bar{v}_x\bar{v}_{xx}}{4\bar{v}^2},
\end{aligned} \tag{48}$$

*then $\bar{V}^{(11)}_2$ and the matrix $\bar{V}_2$ possess the same form as $V^{(11)}_2$ and $V_2$, except changing $u, v$ and their derivatives into $\bar{u}, \bar{v}$ and their derivatives. Thus, the new potentials $\bar{u}$ and $\bar{v}$ are obtained according to the BT (31) from the old ones.*

*Proof.* For $T^{-1} = T^*/\det T$, setting

$$(T_t + TV_2)T^* = \begin{pmatrix} r_{11}(\lambda) & r_{12}(\lambda)\\ r_{21}(\lambda) & r_{22}(\lambda)\end{pmatrix}, \tag{49}$$

one can easily find that the element $r_{jj}(\lambda)$ is $(2N+2)$-th order polynomial, while $r_{jl}(\lambda) (j \neq l)$ is $(2N+1)$-th order one through similar discussion as before. Therefore, we can assume

$$(T_t + TV_2)T^* = (\det T)R(\lambda), \tag{50}$$

i.e.,

$$T_t + TV_2 = R(\lambda)T, \tag{51}$$

with

$$R(\lambda) = \begin{pmatrix} \sum_{l=0}^{3} r^{(l)}_{11}\lambda^l & \sum_{l=0}^{2} r^{(l)}_{12}\lambda^l\\ \sum_{l=0}^{2} r^{(l)}_{21}\lambda^l & \sum_{l=0}^{3} r^{(l)}_{22}\lambda^l\end{pmatrix}, \tag{52}$$

where $r^{(l)}_{kj}$ is independent of $\lambda$.

Considering the first few coefficients of the polynomial of $\lambda$ in (51), we have

$$
\begin{gathered}
r_{11}^{(3)} = -1, \\
r_{22}^{(3)} = 1, \\
r_{11}^{(2)} = r_{22}^{(2)} = 0, \\
r_{21}^{(2)} = -2T_{21}^{N-1} = 2\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right), \\
r_{11}^{(1)} + r_{11}^{(2)} T_{11}^{N-1} + r_{11}^{(3)} T_{11}^{N-2} + r_{12}^{(2)} T_{21}^{N-1} = 2\left(v^3 + \delta\right) - T_{11}^{N-2} + 2\left(v^2 + \frac{\delta}{v}\right) T_{12}^{N-1}, \\
r_{11}^{(2)} T_{12}^{N-1} + r_{11}^{(3)} T_{12}^{N-2} + r_{12}^{(2)} T_{22}^{N-1} = 2uv - v_x + 2vT_{11}^{N-1} + T_{12}^{N-2}, \\
r_{21}^{(1)} + r_{21}^{(2)} T_{11}^{N-1} + r_{22}^{(2)} T_{21}^{N-1} + r_{22}^{(3)} T_{21}^{N-2} = -T_{21}^{N-2} + 2\left(v^2 + \frac{\delta}{v}\right) T_{22}^{N-1}, \\
r_{21}^{(2)} T_{12}^{N-1} + r_{22}^{(2)} T_{22}^{N-1} + r_{22}^{(3)} T_{22}^{N-2} = 2vT_{21}^{N-1} + T_{22}^{N-2}, \\
r_{11}^{(0)} = \left(2v^3 + \delta - r_{11}^{(1)}\right) T_{11}^{N-1} - r_{11}^{(2)} T_{11}^{N-2} - \left(r_{11}^{(3)} + 1\right) T_{11}^{N-3} - r_{12}^{(1)} T_{21}^{N-1} - r_{12}^{(2)} T_{21}^{N-2} + \left[\left(2v - \frac{\delta}{v^2}\right) v_x + 2u\left(v^2 + \frac{\delta}{v}\right)\right] T_{12}^{N-1} + 2\left(v^2 + \frac{\delta}{v}\right) T_{12}^{N-2} + \partial_t(\ln\alpha) + V_2^{(11)}, \\
r_{21}^{(0)} = -r_{21}^{(1)} T_{11}^{N-1} - r_{21}^{(2)} T_{11}^{N-2} - r_{22}^{(1)} T_{21}^{N-1} - r_{22}^{(2)} T_{21}^{N-2} - r_{22}^{(3)} T_{21}^{N-3} + 2\left(v^3 + \delta\right) T_{21}^{N-1} - T_{21}^{N-3} + 2\left(v^2 + \frac{\delta}{v}\right) T_{22}^{N-2} + \left[\left(2v - \frac{\delta}{v^2}\right) v_x + 2u\left(v + \frac{\delta}{v}\right)\right] T_{22}^{N-1}, \\
r_{12}^{(1)} T_{22}^{N-1} = -r_{11}^{(1)} T_{12}^{N-1} - r_{11}^{(2)} T_{12}^{N-2} - r_{11}^{(3)} T_{12}^{N-3} - r_{12}^{(2)} T_{22}^{N-2} + (2uv - v_x) T_{11}^{N-1} + 2vT_{11}^{N-2} - \left(2v^3 + \delta\right) T_{12}^{N-1} + T_{12}^{N-3} + V_2^{(12)}, \\
r_{21}^{(1)} T_{12}^{N-1} = -r_{22}^{(1)} T_{22}^{N-1} - r_{21}^{(2)} T_{12}^{N-2} - r_{22}^{(2)} T_{22}^{N-2} - r_{22}^{(3)} T_{22}^{N-3} + (2uv - v_x) T_{21}^{N-1} + 2vT_{21}^{N-2} - 2\left(v^3 + \delta\right) T_{22}^{N-1} + T_{22}^{N-3}, \\
r_{12}^{(0)} T_{22}^{N-1} = V_2^{(12)} T_{11}^{N-1} + (2uv - v_x) T_{11}^{N-2} + 2vT_{11}^{N-3} + \left(\partial_t \ln\alpha - r_{11}^{(0)} + V_2^{(22)}\right) T_{12}^{N-1} + T_{12,t}^{N-1} - \left(2v^3 + \delta + r_{11}^{(1)}\right) T_{12}^{N-2} - r_{11}^{(2)} T_{12}^{N-3} + \left(1 - r_{11}^{(3)}\right) T_{12}^{N-4} - r_{12}^{(1)} T_{22}^{N-2} - r_{12}^{(2)} T_{22}^{N-3}, \\
r_{22}^{(0)} T_{22}^{N-1} = r_{21}^{(0)} T_{12}^{N-1} - r_{21}^{(1)} T_{12}^{N-2} - r_{21}^{(2)} T_{12}^{N-3} + V_2^{(12)} T_{21}^{N-1} + (2uv - v_x) T_{21}^{N-2} + 2vT_{21}^{N-3} + T_{22,t}^{N-1} + \left(V_2^{(22)} + \partial_t \ln\alpha\right) T_{22}^{N-1} - \left(2v^3 + \delta + r_{22}^{(1)}\right) T_{22}^{N-2} - r_{22}^{(2)} T_{22}^{N-3} + \left(1 - r_{23}^{(2)}\right) T_{22}^{N-4}.
\end{gathered} \tag{53}
$$

On the other side, the coefficients in (35) give

$$
\begin{aligned}
T_{22,x}^{N-2} &= 2\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right) T_{12}^{N-2} - 2vT_{21}^{N-2} - 2(\partial_x \ln\alpha) T_{22}^{N-2}, \\
T_{12,x}^{N-3} &= -2vT_{11}^{N-3} + 2uT_{12}^{N-3} - 2T_{12}^{N-4} + 2\bar{v}T_{22}^{N-3}.
\end{aligned} \tag{54}
$$

Through the similar discussions in Proposition 1 and Proposition 2, we find that

$$
\begin{gathered}
r_{12}^{(0)} = -4\delta\bar{v} + 2\bar{u}^2\bar{v} - 4\bar{v}^4 - \bar{v}\bar{u}_x - 2u\bar{v}_x + \frac{\bar{v}_{xx}}{2}, \\
r_{11}^{(3)} = -r_{22}^{(3)} = -1, \\
r_{11}^{(2)} = r_{22}^{(2)} = 0, \\
r_{12}^{(2)} = 2\bar{v}, \\
r_{21}^{(2)} = 2\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right), \\
r_{11}^{(1)} = -r_{22}^{(1)} = 2\bar{v}^3 + \delta, \\
r_{12}^{(1)} = 2\bar{u}\bar{v} - \bar{v}_x, \\
r_{21}^{(1)} = \left(2\bar{v} - \frac{\delta}{\bar{v}^2}\right)\bar{v}_x + 2\bar{u}\left(\bar{v}^2 + \frac{\delta}{\bar{v}}\right), \\
r_{11}^{(0)} = -r_{22}^{(0)} = -2\delta\bar{u} + \bar{u}^3 - 2\bar{u}\bar{v}^3 - \frac{1}{2}\bar{u}\bar{u}_x + \frac{\delta\bar{u}\bar{u}_x}{\bar{v}^3} - \frac{2\delta\bar{v}_x}{\bar{v}} + \bar{v}^2\bar{v}_x - \frac{\delta\bar{u}_x\bar{v}_x}{2\bar{v}^4} + \frac{\bar{u}_x\bar{v}_x}{\bar{v}} + \frac{\delta\bar{u}\bar{v}_x^2}{2\bar{v}^5} + \frac{\bar{u}\bar{v}_x^2}{2\bar{v}^2} - \frac{\delta\bar{v}_x^3}{4\bar{v}^6} + \frac{\bar{u}_{xx}}{4} - \frac{\delta\bar{u}\bar{v}_{xx}}{2\bar{v}^4} + \frac{\bar{u}\bar{v}_{xx}}{\bar{v}} + \frac{\delta\bar{v}_x\bar{v}_{xx}}{4\bar{v}^5} + \frac{\bar{v}_x\bar{v}_{xx}}{4\bar{v}^2}, \\
r_{21}^{(0)} = -\frac{4\delta^2}{\bar{v}} + \frac{2\delta\bar{u}^2}{\bar{v}} - 8\delta\bar{v}^2 + 2\bar{u}^2\bar{v}^2 - 4\bar{v}^5 + \frac{\delta\bar{u}_x}{\bar{v}} + \bar{v}^2\bar{u}_x - \frac{2\delta\bar{u}\bar{v}_x}{\bar{v}^2} + 4\bar{u}\bar{v}\bar{v}_x + \bar{v}_x^2 + \frac{\delta\bar{v}_x^2}{\bar{v}^3} - \frac{\delta\bar{v}_{xx}}{2\bar{v}^2} + \bar{v}\bar{v}_{xx}.
\end{gathered} \tag{55}
$$

By (18) and (51), we conclude that $\bar{V}_2 = R(\lambda)$. □

According to Proposition 1, Proposition 2, and Proposition 4, based on the transformations (12) and (31), the new Lax pairs (13), (14), and (15) are constructed from the old ones (1), (7), and (9), respectively. By directly computing, both new and old Lax pairs generate the same equations (9) and (10). Then, the following assertion can be gained immediately.

**Theorem 5.** *The potentials $(\bar{u}, \bar{v})$ determined by the DT (12) and (31) are new solutions of equations (9) and (10), respectively.*

## 4. Explicit Solutions and their Application

We will apply the DT of the fractional soliton equations (9), (10) to yield their corresponding soliton solutions in this section. Furthermore, as an application of the yielded solutions, the solitonic solutions of the $(2+1)$-dimensional KP will also be presented.

When $u = \hat{u}$ and $v = \hat{v} \neq 0$ are constants, it is obviously that $(\hat{u}, \hat{v})$ satisfies the soliton equations (9) and (10). Taking it nonzero trivial solution as a seed solution, we can produce nontrivial explicit solutions.

If $\lambda = \lambda_j$, we get two basic solutions of (1), (7), and (9)

$$\begin{aligned} \varphi(\lambda_j) &= \begin{pmatrix} \cosh \xi_j \\ \dfrac{c_j}{2\hat{v}} \sinh \xi_j + \dfrac{\lambda_j - \hat{u}}{2\hat{v}} \cosh \xi_j \end{pmatrix}, \\ \psi(\lambda_j) &= \begin{pmatrix} \sinh \xi_j \\ \dfrac{c_j}{2\hat{v}} \cosh \xi_j + \dfrac{\lambda_j - \hat{u}}{2\hat{v}} \sinh \xi_j \end{pmatrix}, \end{aligned} \tag{56}$$

with $(u, v)^T = (\hat{u}, \hat{v})^T$ is constant vector, where

$$\begin{aligned} \xi_j &= c_j \left\{ x + (\lambda_j + \hat{u}) y + \left[\hat{u}^2 - 2(\hat{v}^3 + \delta)\right] t \right\}, \\ c_j^2 &= (\lambda_j - \hat{u})^2 + 4(\hat{v}^3 + \delta), \quad (j = 1, \cdots, 2N-1). \end{aligned} \tag{57}$$

Taking advantage of (22), we obtain

$$\sigma_j = \frac{c_j}{2v} \frac{1 - \gamma_j \coth \xi_j}{\coth \xi_j - \gamma_j} + \frac{\lambda_j - \hat{u}}{2\hat{v}}, \quad (j = 1, \cdots, 2N-1). \tag{58}$$

According to Cramer's rule, some solutions of (25) and the linear algebraic equations (23) and (24) are

$$\begin{aligned} T_{11}^{N-1} &= \frac{\Delta_{T_{11}^{N-1}}}{\Delta}, \\ T_{21}^{N-1} &= \frac{\hat{v}^3 - (1/2) T_{11,x}^{N-1} + \delta}{\sqrt[3]{\hat{v}^3 - (1/2) T_{11,x}^{N-1}}}, \\ T_{22}^{N-1} &= \frac{\tilde{\Delta}_{T_{22}^{N-1}}}{\tilde{\Delta}}, \\ \tilde{\Delta}_{T_{22}^{N-1}} &= -T_{21}^{N-1} \Delta, \end{aligned} \tag{59}$$

where

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & \sigma_1 & \cdots & \lambda_1^{N-2} & \sigma_1 \lambda_1^{N-2} & \lambda_1^{N-1} \\ 1 & \sigma_2 & \cdots & \lambda_2^{N-2} & \sigma_2 \lambda_1^{N-2} & \lambda_2^{N-1} \\ 1 & \sigma_3 & \cdots & \lambda_3^{N-2} & \sigma_3 \lambda_1^{N-2} & \lambda_3^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N-1} & \cdots & \lambda_{2N-1}^{N-2} & \sigma_{2N-1} \lambda_{2N-1}^{N-2} & \lambda_{2N-1}^{N-1} \end{vmatrix}, \\ \Delta_{T_{11}^{N-1}} &= \begin{vmatrix} 1 & \sigma_1 & \cdots & \lambda_1^{N-2} & \sigma_1 \lambda_1^{N-2} & v\sigma_1 \lambda_1^{N-1} - \lambda_1^N \\ 1 & \sigma_2 & \cdots & \lambda_2^{N-2} & \sigma_2 \lambda_1^{N-2} & v\sigma_2 \lambda_2^{N-1} - \lambda_2^N \\ 1 & \sigma_3 & \cdots & \lambda_3^{N-2} & \sigma_3 \lambda_1^{N-2} & v\sigma_3 \lambda_3^{N-1} - \lambda_3^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N-1} & \cdots & \lambda_{2N-1}^{N-2} & \sigma_{2N-1} \lambda_{2N-1}^{N-2} & v\sigma_{2N-1} \lambda_{2N-1}^{N-1} - \lambda_{2N-1}^N \end{vmatrix}, \\ \tilde{\Delta} &= \begin{vmatrix} 1 & \sigma_1 & \cdots & \lambda_1^{N-2} & \sigma_1 \lambda_1^{N-2} & \sigma_1 \lambda_1^{N-1} \\ 1 & \sigma_2 & \cdots & \lambda_2^{N-2} & \sigma_2 \lambda_1^{N-2} & \sigma_2 \lambda_2^{N-1} \\ 1 & \sigma_3 & \cdots & \lambda_3^{N-2} & \sigma_3 \lambda_1^{N-2} & \sigma_3 \lambda_3^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N-1} & \cdots & \lambda_{2N-1}^{N-2} & \sigma_{2N-1} \lambda_{2N-1}^{N-2} & \sigma_{2N-1} \lambda_{2N-1}^{N-1} \end{vmatrix}. \end{aligned} \tag{60}$$

Therefore, explicit solutions of (9) and (10) are obtained by the BT (31).

$$\begin{cases} \bar{u}[N] = -\dfrac{1}{2} \left( \partial_x \ln \left( T_{22}^{N-1} \right) - 2\hat{u} \right), \\ (\bar{v}[N]) = \sqrt[3]{\hat{v}^3 - \dfrac{1}{2} \partial_x T_{11}^{N-1}}. \end{cases} \tag{61}$$

Next, we will elaborate on the two special cases.

(i) When $N = 1$, let $\lambda = \lambda_1$, for arbitrary constants $\hat{u}$ and $\hat{v} \neq 0$, the solutions of (25) and the linear algebraic equations (23) and (24) are given by

$$\begin{aligned} T_{11}^0 &= \hat{v}\sigma_1 - \lambda_1, \\ T_{22}^0 &= \frac{1}{\sqrt[3]{4}} \frac{\sqrt[3]{2(\hat{v}^3 + \delta) + \hat{v}\sigma_{1,x}^3}}{\sigma_1 \sqrt[3]{2\hat{v}^3 - \hat{v}\sigma_{1,x}}}. \end{aligned} \tag{62}$$

Then, making use of the DT (31) and the Theorem 5, one-soliton solutions of equations (9) and (10) are expressed as

$$\begin{cases} \bar{u}[1] = \hat{u} - \dfrac{6\left(2\hat{v}^3 + \delta - \left(\hat{v}\sigma_x/2\right)\right)\sigma_{1,x}^2 - 12\hat{v}^2\left(\hat{v}^3 + \delta\right)\sigma_{1,x} + 2\left(\delta - 2\hat{v}^3 - \hat{v}\sigma_{1,x}\right)\sigma_1\sigma_{1,xx}}{2\sigma_1\left(6\hat{v}^2 - 3\sigma_{1,x}\right)\left(2\hat{v}^3 + 2\delta - \hat{v}\sigma_{1,x}\right)}, \\ \bar{v}[1] = \sqrt[3]{\hat{v}^3 - \dfrac{1}{2}\sigma_{1,x}} \end{cases} \tag{63}$$

(ii) When $N = 2$, taking $\lambda = \lambda_1, \lambda_2, \lambda_3$, for arbitrary constants $\hat{u}$ and $\hat{v} \neq 0$, the solution of (25) and the linear algebraic equations (23) and (24) are given by

$$\begin{aligned} T_{11}^1 &= \frac{\Delta_{T_{11}^1}}{\Delta}, \\ T_{21}^1 &= \frac{2\hat{v}^3 - T_{11,x}^1 + 2\delta}{\sqrt[3]{8\hat{v}^3 - 4T_{11,x}^1}}, \\ T_{22}^1 &= \frac{\tilde{\Delta}_{T_{22}^1}}{\tilde{\Delta}}, \\ \tilde{\Delta}_{T_{22}^1} &= -T_{21}^1\Delta, \end{aligned} \tag{64}$$

with

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & 1 & 1 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix}, \\ \Delta_{T_{11}^1} &= \begin{vmatrix} 1 & 1 & 1 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \hat{v}\lambda_1\sigma_1 - \lambda_1^2 & \hat{v}\lambda_2\sigma_2 - \lambda_2^2 & \hat{v}\lambda_3\sigma_3 - \lambda_3^2 \end{vmatrix}, \\ \tilde{\Delta} &= \begin{vmatrix} 1 & 1 & 1 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \lambda_1\sigma_1 & \lambda_2\sigma_2 & \lambda_3\sigma_3 \end{vmatrix} \end{aligned} \tag{65}$$

Thus, by the BT (31), we obtain the 3-soliton solutions for equations (9) and (10)

$$\begin{cases} \bar{u}[2] = \hat{u} - \dfrac{1}{2}\partial_x \ln\left(T_{22}^1\right), \\ \left(\bar{v}[2]\right)^3 = \hat{v}^3 - \dfrac{1}{2}\partial_x T_{11}^1. \end{cases} \tag{66}$$

It is well known that the KP equation is a famous $(2+1)$ soliton equation. In the last few years, lump solutions for $(2+1)$ soliton equations have been intense research [38, 39]. Now, let us deduce solutions of the KP equation from the obtained ones. From the following theorem, we can see the relation among them.

**Theorem 6.** *If $u, v$ are solutions of equations (9) and (10), $w(x,y,t) = (v(x,y,t))^3 + \delta$ solves the classic KP equation*

$$w_t = \frac{3}{4}\partial_x^{-1} w_{yy} + \left(\frac{1}{16}w_{xx} - \frac{3}{2}w^2\right)_x. \tag{67}$$

*Proof.* Since $u, v$ satisfy (9) and (10), through directly computing, we conclude that $(v(x,y,t))^3 + \delta$ is a solution of equation (67). □

With the help of Theorem 6, substituting $\bar{v}[N]$ into $\omega$ $(x, y.t)$, we can obtain $N$-soliton solutions of the KP equation as follows

$$\bar{\omega}[N] = \left(\bar{v}[N]\right)^3 + \delta. \tag{68}$$

For example, its 1-soliton solution and 3-soliton solution are $\bar{\omega}[1] = (\bar{v}[1])^3 + \delta$ and $\bar{\omega}[2] = (\bar{v}[2])^3 + \delta$, respectively.

As the parameters are carefully chosen, the figures of the one-soliton solution are plotted in Figure 1. In Figure 2, the spreadings for one-soliton wave packets are plotted for several $t$. In Figure 3, we also draw corresponding images for different eigenvalues.

The 3-dimensional graphs of $u[2], v[2]$, and $w[2]$ are drawn as Figure 4, and their spreadings for several $t$ are plotted in Figure 5.

## 5. Discussion and Conclusion

In the present paper, we investigate a fractional soliton hierarchy which is a special reduction of the D-AKNS system based on the DT method. It is well known that many famous equations with physical meaning can be reduced from the AKNS soliton hierarchy such as the KdV equation, the mKdV equation, and local and nonlocal nonlinear Schrödinger equation. The fractional soliton equation given in the paper is a special case of the D-AKNS hierarchy with $p = 2v, q = 2(v^2 + (\delta/v))$ in (3). Then, we get another reduction of the AKNS hierarchy.

Form the spectral problem with a fractional potential, the $N$-times DT is constructed with the $N$-soliton solution formula given, which has been determined by the linear algebraic system. By choosing proper parameters and avoiding singularities of solutions, we draw several soliton images.

Because of the form of two groups of fundamental solutions of spectral problems, we cannot select the basic solution group as literatures do because the corresponding parameter $\sigma_j$ is an exponential form. Therefore, we can neither simplify the soliton obtained into a simpler form [40] nor conduct a regular analysis [41]. Finally, through a cubic variable transformation, the first two equations in the fractional soliton hierarchy are changed into the $(2+1)$-dimensional KP equation, which provides a method for generating solutions of high-dimensional equations from solutions of low-dimensional equations.

It should be noted that the DT has been proved to be one of most fruitful algorithmic procedures to obtain soliton solutions, so we believe that the DT can be implemented into much more $(2+1)$-dimensional equations in mathematical physics.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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