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Research Article

Sasakian Manifolds Admitting $*$-η-Ricci-Yamabe Solitons

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In this note, we characterize Sasakian manifolds endowed with $*$-η-Ricci-Yamabe solitons. Also, the existence of $*$-η-Ricci-Yamabe solitons in a 5-dimensional Sasakian manifold has been proved through a concrete example.

1. Introduction

In 1982 (resp., 1988), Hamilton introduced the idea of Ricci flow [1] (resp., Yamabe flow [2]). On a smooth Riemannian (or semi-Riemannian manifold), the Yamabe flow is determined as the evolution of the Riemannian (or semi-Riemannian) metric g_0 at time t to $g = g(t)$ using the following equation:

$$\frac{\partial}{\partial t} g(t) = -rg, g(0) = g_0, \tag{1}$$

where $r(t)$ refers to the scalar curvature of the metric $g(t)$. In case $n = 2$, the Yamabe and Ricci flows are related as in the following equation:

$$\frac{\partial}{\partial t} g(t) = -2S(g(t)), \tag{2}$$

where S defines the Ricci tensor. Thus, for the case $n > 2$, there is not such an equivalence, since the Yamabe flow preserves the conformal class of metric but generally this is not true.

The solutions of both Ricci and Yamabe flows are presented as Ricci and Yamabe solitons, respectively. On a Riemannian manifold M, the Ricci and Yamabe solitons are defined by

$$\begin{aligned} &\pounds_F g + 2S + 2\lambda g = 0, \\ &\pounds_F g + 2(\lambda - r) g = 0, \end{aligned} \tag{3}$$

respectively, where $\pounds_F$ is the Lie derivative operator along vector field F (called soliton vector field) at M and $\lambda \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Recently in 2018, Deshmukh and Chen ([3, 4]) briefly studied Yamabe solitons to find sufficient conditions at the soliton vector field so that the metric of the Yamabe soliton is of constant scalar curvature. Yamabe solitons have also been studied in ([5–8]) and many others.

In 2019, Ricci-Yamabe flow, as a new class of geometric flows of the type (α, β), was presented by Güler and Crasmareanu [9] and defined as

$$\frac{\partial}{\partial t} g(t) = \beta r(t) g(t) - 2\alpha S(g(t)), g(0) = g_0. \tag{4}$$

After Güler and Crasmareanu, Dey [10] proposed the concept of Ricci-Yamabe solitons; according to him, the Ricci-Yamabe soliton of the type (α, β) is a Riemannian manifold that admits

$$\frac{1}{2}\pounds_F g + \alpha S + \left(\lambda - \frac{\beta r}{2}\right) g = 0, \tag{5}$$

where $\alpha, \beta \in \mathbb{R}$. In addition, it is noted that Ricci-Yamabe solitons of types $(\alpha, 0)$ and $(0, \beta)$ are known as α-Ricci solitons and β-Yamabe solitons, respectively.

The concept of $*$-Ricci soliton was investigated by Kaimakamis and Panagiotidou [11] in case of real hypersurfaces at complex space forms. More specifically, it is noted that the concept of $*$-Ricci tensor was presented firstly by Tachibana [12] in almost Hermitian manifolds, and later by Hamada [13] to consider different case which is the real hypersurfaces of nonflat complex space forms. The Riemannian metric g on the smooth manifold M is named the $*$-Ricci soliton in case F, a smooth vector field and $\lambda \in \mathbb{R}$ obeying:

$$\frac{1}{2}\pounds_F g = -S^* - g, \tag{6}$$

where

$$S^*(K, L) = g(Q^* K, L) = \text{Trace}\{\varphi \circ R(K, \varphi L)\}, \tag{7}$$

for every vector fields K, L on M, as well as Q^* and S^* are the $*$-Ricci operator and the $*$-Ricci tensor, respectively. In this connection, we recommend the papers ([14–21]) for the specific contents regarding Ricci, η-Ricci, and $*$-Ricci solitons in case of contact Riemannian geometry. In [22], the authors studied gradient Yamabe, gradient Einstein, and quasi-Yamabe solitons on almost co-Kähler manifolds.

Recently, Dey and Roy [23] presented the concept of $*$-η-Ricci soliton in Sasakian manifolds. The Riemannian manifold (M, g) is named $*$-η-Ricci soliton in case

$$\frac{1}{2}\pounds_\zeta g + S^* + \lambda g + \mu \eta \otimes \eta = 0. \tag{8}$$

Motivated by previous studies, we introduce the notion of $*$-η-Ricci-Yamabe soliton of type (α, β) which is a Riemannian manifold satisfying

$$\frac{1}{2}\pounds_F g + \alpha S^* + \left(\lambda - \frac{\beta r}{2}\right) g + \mu \eta \otimes \eta = 0, \tag{9}$$

for $\alpha, \beta, \lambda, \mu \in \mathbb{R}$. The $*$-η-Ricci-Yamabe soliton is described as shrinking, steady or expanding if it admits the soliton vector for which $\lambda < 0$, $= 0$ or >0, respectively. Particularly, if $\mu = 0$, then this concept of $*$-η-Ricci-Yamabe soliton $(g, F, \lambda, \mu, \alpha, \beta)$ reduces to a concept of $*$-Ricci-Yamabe soliton $(g, F, \lambda, \alpha, \beta)$.

Throughout the paper, we denote a $(2n+1)$-dimensional Sasakian manifold by M^S_{2n+1}, $*$-Ricci-Yamabe soliton by $*$-RYS, and $*$-η-Ricci-Yamabe soliton by $*$-η-RYS. We present our work as follows: Section 2 includes essential results and some basic definitions of Sasakian manifolds. Section 3 covers the study of $*$-η-RYS on M^S_{2n+1} leading to several significant characterizations of the manifold. Section 4 deals with the study of pseudo-Ricci-symmetric and Ricci recurrent M^S_{2n+1} admitting $*$-η-RYS. The $*$-η-RYS on M^S_{2n+1} satisfying the curvature conditions $R(\zeta, X) \cdot S = 0$ and $Q(g, S) = 0$ have been studied in Sections 5 and 6, respectively.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is said to admit an almost contact structure, sometimes called a (φ, ζ, η)-structure, in case it admits a (1,1) type tensor field φ, a structure vector field ζ, and a 1-form η satisfying [24]

$$\varphi^2 = -I + \eta \otimes \zeta, \eta(\zeta) = 1, \varphi\zeta = 0, \eta \circ \varphi = 0. \tag{10}$$

The almost contact structure is called normal in case $\aleph + d\eta \otimes \zeta = 0$, where $\aleph$ is the Nijenhuis tensor of φ. Considering the Riemannian metric tensor g that is defined on M and satisfies

$$g(\varphi K, \varphi L) = g(K, L) - \eta(K)\eta(L), \eta(K) = g(K, \zeta), \tag{11}$$

for any $K, L \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ refers to the set of all smooth vector fields of M. The structure $(\varphi, \zeta, \eta, g)$ is named the almost contact metric structure. Next, considering Φ, the tensor field of type $(0, 2)$ as $\Phi(K, L) = g(\Phi K, L)$. In case $d\eta = \Phi$, then the structure $(\varphi, \zeta, \eta, g)$ is named as normal metric structure. The normal contact metric structure is named Sasakian structure satisfying ([25–27]):

$$(\nabla_K \varphi)L = g(K, L)\zeta - \eta(L)K, \tag{12}$$

for any $K, L \in \mathfrak{X}(M)$, where ∇ stands for the Levi-Civita connection.

In case of M^S_{2n+1}, we have

$$R(\zeta, K)L = g(K, L)\zeta - \eta(L)K, \tag{13}$$

$$R(K, L)\zeta = \eta(L)K - \eta(K)L, \tag{14}$$

$$S(K, \zeta) = 2n\eta(K) \Leftarrow Q\zeta = 2n\zeta, \tag{15}$$

$$\nabla_K \zeta = -\varphi K, \tag{16}$$

$$(\nabla_K \eta)L = -g(\varphi K, L), \tag{17}$$

for any $K, L \in \mathfrak{X}(M)$; R and Q refers to the curvature tensor and the Ricci operator.

Definition 1. A Sasakian manifold is called an η-Einstein in case the non-vanishing Ricci tensor S is expressed as

$$S(K, L) = a g(K, L) + b\eta(K)\eta(L), \tag{18}$$

where $a, b \in C^\infty(M)$. In particular, if $b = 0$, then M is named as an Einstein manifold.

Definition 2. The vector field V is named as an affine conformal vector field in case it satisfies [28]

$$(\pounds_V \nabla)(K, L) = L(\rho)K + K(\rho)L - g(K, L) grad\rho, \tag{19}$$

where $\rho \in C^{\infty}(M)$. In case $\rho \in \mathbb{R}$, then V is called an affine vector field.

Lemma 3. *The $*$ -Ricci tensor of M^S_{2n+1} is given by [14]*

$$S^*(K, L) = S(K, L) - (2n-1)g(K, L) - \eta(K)\eta(L), \tag{20}$$

for any $K, L \in \mathfrak{X}(M)$.

3. $*$-η-Ricci-Yamabe Solitons on Sasakian Manifolds

First, we prove the following:

Theorem 4. *An M^S_{2n+1} admitting $*$ - η -RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$ is an η -Einstein manifold of the constant scalar curvature. Moreover, the scalars λ, μ related to each other by $\lambda + \mu = \beta$ $r/2$.*

Proof. Let the metric of an M^S_{2n+1} be $*$-η-RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$, then Equation (9) turns to

$$g(\nabla_K \zeta, L) + g(K, \nabla_L \zeta) + 2\alpha S^*(K, L) + (2\lambda - \beta r)g(K, L) + 2\mu\eta(K)\eta(L) = 0, \tag{21}$$

for all vector fields K as well L on M. Using (16), Equation (21) leads to

$$S^*(K, L) = -\frac{1}{\alpha}\left(\lambda - \frac{\beta r}{2}\right)g(K, L) - \frac{\mu}{\alpha}\eta(K)\eta(L), \alpha \neq 0. \tag{22}$$

Using (20), (22) takes the form

$$S(K, L) = \sigma_1 g(K, L) + \sigma_2 \eta(K)\eta(L), \tag{23}$$

where $\sigma_1 = 2n - 1 - (1/\alpha)(\lambda - (\beta r/2))$ and $\sigma_2 = 1 - (\mu/\alpha)$.

By putting $L = \zeta$ at (23) as well the use of (10) and (11), we have

$$S(K, \zeta) = (\sigma_1 + \sigma_2)\eta(K), \tag{24}$$

where $\sigma_1 + \sigma_2 = 2n - (1/\alpha)(\lambda + \mu - (\beta r/2))$.

In view of (15), from (24), it follows that

$$\lambda + \mu = \frac{\beta r}{2}, \text{where } \alpha \neq 0. \tag{25}$$

On contracting (23), we find $r = \sigma_1(2n+1) + \sigma_2$, which by using the values of σ_1, σ_2 and (25) leads to

$$r = 2n\left(2n + \frac{\mu}{\alpha}\right), \tag{26}$$

where μ and $\alpha(\neq 0)$ are constants. Thus, (23) together with (25) and (26) leads to the statement of Theorem 4. □

Particularly, taking $\mu = 0$ in (23) as well in (25) resulted in $S(K, L) = (2n-1)g(K, L) + \eta(K)\eta(L)$ and $\lambda = 2n^2\beta$, respectively, being $r = 4n^2$. Thus, we have the following.

Corollary 5. *An M^S_{2n+1} admitting $*$ -RYS $(g, \zeta, \lambda, \alpha, \beta)$ is an η -Einstein manifold, and the soliton is shrinking, steady or expanding according to $\beta < 0$, $= 0$ or >0, respectively.*

Next, we prove the following.

Theorem 6. *If an M^S_{2n+1} admits $*$-η-RYS $(g, F, \lambda, \mu, \alpha, \beta)$ such that the vector field F represents an affine conformal vector field. Then, M^S_{2n+1} is an η-Einstein manifold, and F is an affine vector field.*

Proof. The use of (20) in (9) gives

$$\begin{aligned}(\pounds_F g)(L, U) = &-2\alpha S(L, U) + [2\alpha(2n-1) - (2\lambda - \beta r)]g(L, U) \\ &+ 2(\alpha - \mu)\eta(L)\eta(U).\end{aligned} \tag{27}$$

Referencing Yano [29], the expression

$$\begin{aligned}&\left(\pounds_F \nabla_K g - \nabla_K \pounds_F g - \nabla_{[F,K]} g\right)(L, U) \\ &= -g((\pounds_F \nabla)(K, L), U) - g((\pounds_F \nabla)(K, U), L),\end{aligned} \tag{28}$$

is well-known for all K, L, U at M. As g is parallel respecting to ∇, the previous equation turns to

$$(\nabla_K \pounds_F g)(L, U) = g((\pounds_F \nabla)(K, U), L) + g((\pounds_F \nabla)(K, L), U), \tag{29}$$

as a result of (19), it leads to

$$(\nabla_K \pounds_F g)(L, U) = 2K(\rho)g(L, U). \tag{30}$$

Taking the covariant derivative of (27) respecting to K and using (17), we have

$$\begin{aligned}(\nabla_K \pounds_F g)(L, U) = &-2\alpha(\nabla_K S)(L, U) + \beta K(r)g(L, U) \\ &- 2(\alpha - \mu)(g(\varphi K, L)\eta(U) + g(\varphi K, U)\eta(L)).\end{aligned} \tag{31}$$

Putting $L = U = \zeta$ in (31) and using (10), (11), (15), and (30), we get

$$2K(\rho) = \beta K(r). \tag{32}$$

From (30)–(32), we find

$$\alpha(\nabla_K S)(L, U) + (\alpha - \mu)(g(\varphi K, L)\eta(U) + g(\varphi K, U)\eta(L)) = 0, \tag{33}$$

which by replacing $U = \zeta$ gives

$$(\nabla_K S)(L, \zeta) = \left(\frac{\mu}{\alpha} - 1\right) g(\varphi K, L), \alpha \neq 0. \tag{34}$$

Now, the covariant differentiation of (15) yields

$$(\nabla_K S)(L, \zeta) = S(L, \varphi K) - 2ng(\varphi K, L). \tag{35}$$

From (34) and (35), it follows that

$$S(L, \varphi K) = \left(2n - 1 + \frac{\mu}{\alpha}\right) g(\varphi K, L). \tag{36}$$

By replacing K by φK in (36) and using (10), we get

$$S(K, L) = \left(2n - 1 + \frac{\mu}{\alpha}\right) g(K, L) - \left(\frac{\mu}{\alpha} - 1\right) \eta(K)\eta(L), \alpha \neq 0. \tag{37}$$

The contraction of (37) gives $r = 2n(2n + \mu/\alpha)$. Therefore, from (32), it follows that $K(\rho) = 0$. This implies that $\rho \in \mathbb{R}$; therefore, F is an affine vector field. This completes the proof. □

Furthermore, we prove the following.

Lemma 7. *An M^S_{2n+1} satisfies the following equations:*

$$(\nabla_L Q)\zeta = Q\varphi L - 2n\varphi L, \tag{38}$$

$$\left(\nabla_\zeta Q\right) L = 2Q\varphi L, \tag{39}$$

where Q refers to the Ricci operator.

Proof. Differentiating $Q\zeta = 2n\zeta$ along L and using (16), we get (38). Next, differentiating (14) along W and using (16), we find

$$(\nabla_W R)(K, L)\zeta = R(K, L)\varphi W - g(\varphi W, L)K + g(\varphi W, K)L. \tag{40}$$

Taking a frame field and then contracting (40), we get

$$\sum_{i=1}^{2n+1} g\left(\left(\nabla_{e_i} R\right)(e_i, L)\zeta, U\right) = -S(L, \varphi U) + 2ng(\varphi L, U). \tag{41}$$

From Bianchi's second identity, we can easily obtain that

$$\sum_{i=1}^{2n+1} g\left(\left(\nabla_{e_i} R\right)(e_i, L)\zeta, U\right) = (\nabla_U S)(\zeta, L) - \left(\nabla_\zeta S\right)(U, L). \tag{42}$$

By equating (41) and (42), then using (38), Equation (39) follows. □

Now, we prove the next theorem:

Theorem 8. *If an M^S_{2n+1} admits $*$ - η -RYS $(g, F, \lambda, \mu, \alpha, \beta)$ such that the vector field F represents the gradient Dr of r defined by (9), then either F is a pointwise collinear with the structure vector field ζ or $\beta = -2$.*

Proof. Suppose an M^S_{2n+1} admits $*$-η-RYS $(g, F, \lambda, \mu, \alpha, \beta)$ such that the vector field F represents the gradient Dr of r, i.e., $F = Dr$. Then, from (9), we find

$$\nabla_K Dr = -\alpha QK - \left(\lambda - \frac{\beta r}{2} - \alpha(2n - 1)\right)K + (\alpha - \mu)\eta(K)\zeta, \tag{43}$$

for any K on M.

The covariant differentiation of (43) respecting to L and the use of (16) and (17) leads to

$$\begin{aligned}\nabla_L \nabla_K Dr = {} & -\alpha((\nabla_L Q)K + Q(\nabla_L K)) - \left(\lambda - \frac{\beta r}{2} - \alpha(2n - 1)\right)\nabla_L K \\ & + \frac{\beta}{2} L(r)K + (\alpha - \mu)(-g(\varphi K, L)\zeta + \eta(\nabla_L K)\zeta - \eta(K)\varphi L).\end{aligned} \tag{44}$$

Interchanging K and L in (44), we have

$$\begin{aligned}\nabla_K \nabla_L Dr = {} & -\alpha((\nabla_K Q)L + Q(\nabla_K L)) - \left(\lambda - \frac{\beta r}{2} - \alpha(2n - 1)\right)\nabla_K L \\ & + \frac{\beta}{2} K(r)L + (\alpha - \mu)(-g(\varphi L, K)\zeta + \eta(\nabla_K L)\zeta - \eta(L)\varphi K).\end{aligned} \tag{45}$$

In view of (43), we also have

$$\begin{aligned}\nabla_{[K,L]} Dr = {} & -\alpha Q(\nabla_K L) + \alpha Q(\nabla_L K) - \left(\lambda - \frac{\beta r}{2} - \alpha(2n - 1)\right)\nabla_K L \\ & + \left(\lambda - \frac{\beta r}{2} - \alpha(2n - 1)\right)\nabla_L K + (\alpha - \mu)(\eta(\nabla_K L)\zeta - \eta(\nabla_L K)\zeta).\end{aligned} \tag{46}$$

From (44)–(46), we get

$$\begin{aligned}R(K, L)Dr = {} & \alpha((\nabla_L Q)K - (\nabla_K Q)L) + \frac{\beta}{2}(K(r)L - L(r)K) \\ & + (\alpha - \mu)(2g(K, \varphi L)\zeta + \eta(K)\varphi L - \eta(L)\varphi K).\end{aligned} \tag{47}$$

By replacing K by ζ in (47) and using (10), (13), (38), and (39), we get

$$L(r)\zeta - \zeta(r)L = -\alpha(Q\varphi L + 2n\varphi L) + \frac{\beta}{2}(\zeta(r)L - L(r)\zeta) + (\alpha - \mu)\varphi L. \tag{48}$$

The inner product of (48) with ζ leads to

$$\left(1 + \frac{\beta}{2}\right)(L(r) - \xi(r)\eta(L)) = 0. \tag{49}$$

Therefore, we have either $\beta = -2$ or $F = Dr = \xi(r)\xi$, that is, F is pointwise collinear with ζ. The proof is completed. □

4. Pseudo-Ricci-Symmetric and Ricci-Recurrent Sasakian Manifolds Admitting $*$-η-Ricci-Yamabe Solitons

Definition 9. The non-flatM^S_{2n+1} is named pseudo-Ricci-symmetric and is represented by $(PRS)_{2n+1}$, in case the Ricci tensor $S(\neq 0)$ of the manifold satisfies the condition [30]

$$(\nabla_U S)(K, L) = 2A(U)S(K, L) + A(K)S(U, L) + A(L)S(U, K), \tag{50}$$

where the non-zero 1-form A is given by $g(U, \zeta) = A(U)$, $\forall$ vector fields U ; ζ being the vector field that corresponds to the associated 1-form A. In particular, if $A = 0$, then M^S_{2n+1} is called Ricci-symmetric.

The covariant derivative of (23) leads to

$$(\nabla_U S)(K, L) = \sigma_2[g(U, \varphi K)\eta(L) + g(U, \varphi L)\eta(K)]. \tag{51}$$

Now, using (23) and (51), (50) becomes

$$\begin{aligned}\sigma_2[g(U, \varphi K)\eta(L) + g(U, \varphi L)\eta(K)] \\ = 2A(U)[\sigma_1 g(K, L) + \sigma_2\eta(K)\eta(L)] \\ + A(K)[\sigma_1 g(U, L) + \sigma_2\eta(U)\eta(L)] \\ + A(L)[\sigma_1 g(U, K) + \sigma_2\eta(U)\eta(K)].\end{aligned} \tag{52}$$

Choosing $U = L = \zeta$, (52) reduces to $A(K) = -3A(\zeta)\eta(K)$ which by putting $K = \zeta$ gives $A(\zeta) = 0$. This implies that $A(K) = 0$. Thus, we have the following.

Theorem 10. *A pseudo-Ricci-symmetric M^S_{2n+1} admitting $*$ - η -RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$ is Ricci-symmetric.*

Definition 11 [31]. An M^S_{2n+1} is named as Ricci-recurrent in case there exists a 1-form $\omega(\neq 0)$ holds:

$$(\nabla_K S)(L, U) = \omega(K)S(L, U). \tag{53}$$

for all K, L and U on M and 1-form ω.

By the use of (51) in (53), we find

$$\sigma_2[g(K, \varphi L)\eta(U) + g(K, \varphi U)\eta(L)] = \omega(K)S(L, U), \tag{54}$$

which by putting $U = \zeta$ then using (10) and (15) reduces to

$$\sigma_2 g(K, \varphi L) = 2n\omega(K)\eta(L). \tag{55}$$

By taking $\omega = \eta$, (55) takes the form

$$\sigma_2 g(K, \varphi L) = 2n\eta(K)\eta(L). \tag{56}$$

Now, replacing K by φK in (56) and using (10), we find

$$\sigma_2 g(\varphi K, \varphi L) = 0. \tag{57}$$

Since $g(\varphi K, \varphi L) \neq 0$, therefore, we obtain $\sigma_2 = 0$. This leads to $\mu = \alpha$. Hence, by the use of (25), we have $\lambda = -\alpha + \beta r/2$. Therefore, we give the next theorem.

Theorem 12. *If a Ricci-recurrent M^S_{2n+1} admits $*$ - η -RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$, then $\lambda = -\alpha + \beta r/2$ as well $\mu = \alpha$.*

Hence, by using these values of λ and μ in (23), we obtain

$$S(K, L) = 2ng(K, L). \tag{58}$$

Thus, we state:

Corollary 13. *A Ricci-recurrent M^S_{2n+1} admitting a $*$ - η -RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$ defines an Einstein manifold.*

5. Sasakian Manifolds Admitting $*$-η-Ricci-Yamabe Solitons Satisfying $R(\zeta, X) \cdot S = 0$

Considering an M^S_{2n+1} admitting $*$-η-RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$ which satisfies $R(\zeta, X) \cdot S = 0$, this implies that

$$S(R(\zeta, K)L, U) + S(L, R(\zeta, K)U) = 0, \tag{59}$$

for all K, L, U on M. In view of (23) and the symmetries of R , (59) takes the form

$$\sigma_2(g(K, L)\eta(U) + g(K, U)\eta(L) - 2\eta(K)\eta(L)\eta(U)) = 0, \tag{60}$$

which by taking $U = \zeta$ then using (10) and (11) turns to

$$\sigma_2 g(\varphi K, \varphi L) = 0. \tag{61}$$

From (61), it follows that $\sigma_2 = 0$, which leads to $\mu = \alpha$; hence, (25) gives $\lambda = \beta r/2 - \alpha$. This helps us to state:

Theorem 14. *For an M^S_{2n+1} admitting $*$ - η -RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$ that satisfies $R(\zeta, X) \cdot S = 0$, we have $\lambda = -\alpha + \beta r/2$ and $\mu = \alpha$.*

Now by using $\lambda = -\alpha + \beta r/2$ and $\mu = \alpha$, (23) takes the form

$$S(K, L) = 2ng(K, L). \tag{62}$$

Thus, we have:

Corollary 15. *In case an M^S_{2n+1} satisfies $R(\zeta, X) \cdot S = 0$ and admits $*$ - η -RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$, then it defines an Einstein manifold.*

6. Sasakian Manifolds Admitting $*$-η-Ricci-Yamabe Solitons Satisfying $Q(g, S) = 0$

Let an M^S_{2n+1} admitting $*$-η-RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$ satisfies

$$Q(g, S)(K, L, U, W) = 0, \tag{63}$$

where

$$Q(g, S)(K, L, U, W) = \left(\left(K_{\wedge g}L\right) \cdot S\right)(U, W). \tag{64}$$

This can be expressed as

$$\begin{aligned} Q(g, S)(K, L, U, W) = g(L, U)S(K, W) - g(K, U)S(L, W) \\ + g(L, W)S(K, U) - g(K, W)S(L, U), \end{aligned} \tag{65}$$

where $(K_{\wedge g}L)U = g(L, U)K - g(K, U)L$ being used.

From (63), (65), and (23), we get

$$\begin{aligned} &\sigma_2(g(L, U)\eta(K)\eta(W) - g(K, U)\eta(L)\eta(W) \\ &+ g(L, W)\eta(K)\eta(U) - g(K, W)\eta(L)\eta(U)) = 0. \end{aligned} \tag{66}$$

From the preceeding equation, it follows that $\sigma_2 = 0$. This implies that $\mu = \alpha$. Hence, from (25), we get $\lambda = -\alpha + \beta r/2$. Thus, we have

Theorem 16. *If an M^S_{2n+1} admits $*$ - η -RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$ and the manifold satisfies $Q(g, S) = 0$, then $\lambda = -\alpha + \beta r/2$ and $\mu = \alpha$.*

Now, by using these values of λ as well μ, (23) yields

$$S(K, L) = 2ng(K, L). \tag{67}$$

Thus, we give the next corollary:

Corollary 17. *In case an M^S_{2n+1} admitting $*$ - η -RYS $(g, \zeta, \lambda, \mu, \alpha, \beta)$ satisfies $Q(g, S) = 0$, then it is an Einstein manifold.*

Example 1. Let a manifold $M = \{(u, v, w, s, t) \in \mathbb{R}^5\}$ of dimension 5, where (u, v, w, s, t) refer to the usual coordinates at $\mathbb{R}^5$. Suppose ρ_1,ρ_2, ρ_3, ρ_4, and ρ_5 are the vector fields at M defined as

$$\rho_1 = \frac{\partial}{\partial u}, \rho_2 = \left(\frac{\partial}{\partial w} - 2u\frac{\partial}{\partial t}\right), \rho_3 = \frac{\partial}{\partial v}, \rho_4 = \left(\frac{\partial}{\partial s} - 2v\frac{\partial}{\partial t}\right), \rho_5 = \frac{\partial}{\partial t} = \zeta, \tag{68}$$

and these are linearly independent at each point of M.

Suppose g is the Riemannian metric defined as

$$\begin{aligned} g\left(\rho_i, \rho_j\right) = 0, 1 \le i \ne j \le 5, \\ g\left(\rho_i, \rho_j\right) = 1, 1 \le i = j \le 5. \end{aligned} \tag{69}$$

Considering η, a 1-form on M determined as $\eta(K) = g(K, \rho_5) = g(K, \zeta)$ of all $K \in \chi(M)$. Let φ be a $(1, 1)$ tensor field on M defined by

$$\varphi\rho_1 = -\rho_2, \varphi\rho_2 = \rho_1, \varphi\rho_3 = -\rho_4, \varphi\rho_4 = \rho_3, \varphi\rho_5 = 0. \tag{70}$$

The linearity of φ and g leads to

$$\begin{aligned} \eta(\zeta) = 1, \varphi^2 K = -K + \eta(K)\zeta, \eta(\varphi K) = 0, \\ g(K, \zeta) = \eta(K), g(\varphi K, \varphi L) = g(K, L) - \eta(K)\eta(L), \end{aligned} \tag{71}$$

for all $K, L \in \chi(M)$. Therefore $[\rho_1, \rho_2] = 2\rho_5, [\rho_3, \rho_4] = -2\rho_5$ and $[\rho_i, \rho_j] = 0$ for others i and j. By using well-known Koszul's formula, we can easily calculate

$$\begin{aligned} &\nabla_{\rho_1}\rho_1 = 0, \nabla_{\rho_1}\rho_2 = -\rho_5, \nabla_{\rho_1}\rho_3 = 0, \nabla_{\rho_1}\rho_4 = 0, \nabla_{\rho_1}\rho_5 = \rho_2, \\ &\nabla_{\rho_2}\rho_1 = \rho_5, \nabla_{\rho_2}\rho_2 = 0, \nabla_{\rho_2}\rho_3 = 0, \nabla_{\rho_2}\rho_4 = 0, \nabla_{\rho_2}\rho_5 = -\rho_1, \\ &\nabla_{\rho_3}\rho_1 = 0, \nabla_{\rho_3}\rho_2 = 0, \nabla_{\rho_3}\rho_3 = 0, \nabla_{\rho_3}\rho_4 = -\rho_5, \nabla_{\rho_3}\rho_5 = \rho_4, \\ &\nabla_{\rho_4}\rho_1 = 0, \nabla_{\rho_4}\rho_2 = 0, \nabla_{\rho_4}\rho_3 = \rho_5, \nabla_{\rho_4}\rho_4 = 0, \nabla_{\rho_4}\rho_5 = -\rho_3, \\ &\nabla_{\rho_5}\rho_1 = \rho_2, \nabla_{\rho_5}\rho_2 = -\rho_1, \nabla_{\rho_5}\rho_3 = \rho_4, \nabla_{\rho_5}\rho_4 = -\rho_3, \nabla_{\rho_5}\rho_5 = 0. \end{aligned} \tag{72}$$

It can be easily verified that the manifold satisfies

$$\nabla_K\zeta = -\varphi K \text{ and } (\nabla_K\varphi)L = g(K, L)\zeta - \eta(L)K \text{ for } \zeta = \rho_5. \tag{73}$$

It is clear that this manifold M is a Sasakian manifold.

It is easy to have the following non-vanishing components:

$$\begin{aligned} &R(\rho_1, \rho_2)\rho_1 = 3\rho_2, R(\rho_1, \rho_5)\rho_1 = -\rho_5, R(\rho_1, \rho_2)\rho_2 \\ &\quad = -3\rho_1, R(\rho_2, \rho_5)\rho_2 = -\rho_5, \\ &R(\rho_3, \rho_4)\rho_3 = 3\rho_4, R(\rho_3, \rho_5)\rho_3 = -\rho_5, R(\rho_3, \rho_4)\rho_4 \\ &\quad = -3\rho_3, R(\rho_4, \rho_5)\rho_4 = -\rho_5, \\ &R(\rho_1, \rho_5)\rho_5 = \rho_1, R(\rho_2, \rho_5)\rho_5 = \rho_4, R(\rho_3, \rho_5)\rho_5 \\ &\quad = \rho_3, R(\rho_4, \rho_5)\rho_5 = \rho_4. \end{aligned} \tag{74}$$

Utilizing the previous results we calculate the following:

$$S(\rho_1, \rho_1) = S(\rho_2, \rho_2) = S(\rho_3, \rho_3) = S(\rho_4, \rho_4) = -2, S(\rho_5, \rho_5) = 4. \tag{75}$$

Using (23), we have $S(\rho_5, \rho_5) = 4 - 1/\alpha(\lambda + \mu - \beta r/2)$. By equating both the values of $S(\rho_5, \rho_5)$, we obtain

$$\lambda + \mu = \frac{\beta r}{2}, \alpha \neq 0. \tag{76}$$

Hence, λ as well μ insures Equation (25), and so, g is the $*$-η-RYS on the given 5-dimensional Sasakian manifold.

Data Availability

No data were used to support this study.

Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Authors' Contributions

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