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Some Commutativity Theorems in Prime Rings with Involution and Derivations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Let R be a ring with involution $'*'$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. An additive mapping $\delta : R \to R$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. The purpose of this paper is to examine the commutativity of prime rings with involution satisfying certain identities involving derivations.

Keywords: *Prime ring; normal ring; commutativity; involution; derivation.*

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1 Introduction and Notations

Throughout this paper, R always denotes an associative ring with centre $Z(R)$. As usual the symbols $s \circ t$ and $[s, t]$ will denote the anti-commutator $st + ts$ and commutator $st - ts$, respectively. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if $nx = 0$ (where $x \in R$) implies that $x = 0$. A ring R is called prime if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$, and is called semiprime ring if $aRa = (0)$ (where $a \in R$) implies $a = 0$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) and (ii) $(x^*)^* = x$ hold for all $x, y \in R$. A ring equipped with an involution is called ring with involution or $*$-ring. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is called the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq (0)$. Notice that in case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring(see [1] for more details). An additive mapping $\delta : R \to R$ is said to be a derivation of R if $\delta(st) = \delta(s)t + s\delta(t)$ for all $s, t \in R$. A derivation δ is said to be inner if there exists $a \in R$ such that $\delta(s) = as - sa$ for all $s \in R$. Over the last some decades, several authors have investigated the relationship between the commutativity of the ring R and certain special types of maps like derivations and automorphisms of R. The criteria to discuss the commutativity of certain classes of rings via derivations had been given first time by Posner [2]. In fact, proved that the existence of a nonzero centralizing derivation (*i.e.*, $\delta(x)x - x\delta(x) \in Z(R)$ for all $x \in R$) on a prime ring forces the ring to be commutative. Since then many algebraists established the commutativity of prime and semiprime rings via derivations or automorphisms that satisfying certain identities (viz.; [3], [4], [5], [6], [7] [8], [9], [10], [11], [12] and references therein). In [13], Bell and Daif showed that if R is a prime ring admitting a nonzero derivation δ such that $\delta(st) = \delta(ts)$ for all $s, t \in R$, then R is commutative. This result was extended for semiprime rings by Daif [14]. In 2016, S. Ali et. al [15], studied these results in the setting of rings with involution involving derivations (see also [16]). In this paper, our intent is to continue this line of investigation and to discuss the commutativity of prime rings with involution involving derivations in more general situation.

2 The Results

We start our investigation with some well known facts and results in rings which will be used frequently throughout the text.

Fact 2.1 ([18, Lemma 2.1])**.** *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. If R is normal i.e., $[x, x^*] = 0$ for all $x \in R$, then R is commutative.*

Fact 2.2. *The center of a prime ring is free from zero divisors.*

Fact 2.3. *Let R be a 2-torsion free ring with involution $'*'$. Then every $x \in R$ can be uniquely represented as $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.*

In [17], the authors did not stated Lemma 2.1 correctly. The correct statement is the following.

Fact 2.4. *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. If $[x, x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Fact 2.5. *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Let δ be a derivation of R such that $\delta(h) = 0$ for all $h \in H(R) \cap Z(R)$. Then $\delta(x) = 0$ for all $x \in R$.*

Proof. By the assumption, we have $\delta(h) = 0$ for all $h \in H(R) \cap Z(R)$. Substituting k^2(where $k \in S(R) \cap Z(R)$) for h and using the fact that $\delta(k) \in Z(R)$, we obtain $2\delta(k)k = 0$ for all $k \in S(R) \cap Z(R)$. This implies that $\delta(k)k = 0$ for all $k \in S(R) \cap Z(R). Application of Fact 2.2 yields \delta(k) = 0$ for all $k \in S(R) \cap Z(R)$. In view of Fact 2.3, we conclude that $2\delta(x) = \delta(2x) = \delta(h + k) = \delta(h) + \delta(k) = 0$ and hence $\delta(x) = 0$ for all $x \in R$. □

In [18], first author together with N. A. Dar proved the following theorem.

Theorem 2.1. *Let R be a prime ring with involution $'*'$ such that $char(R) \neq 2$. Let δ be a nonzero derivation of R such that $\delta([x, x^*]) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then R is commutative.*

In the following theorem, we prove the same result in a more general setting.

Theorem 2.2. *Let R be a prime ring with involution$'*'$ of the second kind such that $char(R) \neq 2$. Let δ be a nonzero derivation of R such that $\delta([x, x^*]) \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. By the hypothesis, we have

$$\delta([x, x^*]) \in Z(R) \tag{2.1}$$

for all $x \in R$. Substituting x by $x + y$ in (2.1), we obtain

$$\delta([x, y^*]) + \delta([y, x^*]) + \in Z(R) \tag{2.2}$$

for all $x, y \in R$. Replacing y by yh (where $h \in Z(R) \cap H(R)$ in (2.2), we get

$$\delta(h)[x, y^*] + h\delta([x, y^*]) + \delta([y, x^*])h + [y, x^*]\delta(h) \in Z(R) \tag{2.3}$$

for all $x, y \in R$. Since $h \in Z(R) \cap H(R)$ and δ is nonzero derivation of R, last expression can be written as

$$([x, y^*] + [y, x^*])\delta(h) + h(\delta([x, y^*]) + \delta([y, x^*])) \in Z(R) \tag{2.4}$$

for all $x, y \in R$. Applications of (2.2) yields that

$$([x, y^*] + [y, x^*])\delta(h) \in Z(R) \tag{2.5}$$

for all $x, y \in R$. Taking $x = y$ in (2.5), we arrive at

$$2[x, x^*]\delta(h) \in Z(R) \tag{2.6}$$

for all $x \in R$. Since $char(R) \neq 2$, so the last relation gives $[x, x^*]\delta(h) \in Z(R)$ for all $x \in R$. It is well known that if R is prime and $0 \neq t \in Z(R)$ such that $xt \in Z(R)$, then $x \in Z(R)$. Thus, we conclude that either $[x, x^*] \in Z(R)$ for all $x \in R$ or $\delta(h) = 0$ for all $h \in Z(R) \cap H(R)$. If $\delta(h) = 0$ for all $h \in Z(R) \bigcap H(R)$. Replacing h by k^2 (where $k \in S(R) \cap Z(R)$ in the last expression, we get $2\delta(k)k = 0$ for all $k \in S(R) \cap Z(R)$. Since $char(R) \neq 2$, we arrive at $\delta(k)k = 0$ for all $k \in S(R) \cap Z(R)$. Since $k \in S(R) \cap Z(R)$ and R is prime, so by Fact 2.2 we conclude that $\delta(k) = 0$ for all $k \in S(R) \cap Z(R)$. In view Fact 2.3, for every $x \in R$, we write $2x = h + k$, where $h \in H(R),\ k \in S(R)$ and hence we conclude by Fact 2.5 that $\delta(x) = 0$ for all $x \in R$, a contradiction. Consequently, we have $[x, x^*] \in Z(R)$ for all $x \in R$. Therefore, application of Fact 2.4 yields the required conclusion. Hence, R is commutative. This completes the proof of the theorem. □

We now prove the anti-commutator version of Theorem 2.2.

Theorem 2.3. *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Let δ be a nonzero derivation of R such that $\delta(x \circ x^*) \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. A careful scrutiny shows that the proof runs on parallel lines as in Theorem 2.2 and hence we skip the details of proof just to avoid repetition. □

As consequences of Theorem 2.2 and Theorem 2.3, we obtain the two main results of [15].

Corollary 2.1 ([15, Theorem 2.2]). *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Let δ be a nonzero derivation of R such that $\delta([x, x^*]) = (0)$ for all $x \in R$. Then R is commutative.*

Corollary 2.2 ([15, Theorem 2.3]). *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Let δ be a nonzero derivation of R such that $\delta(x \circ x^*) = (0)$ for all $x \in R$. Then R is commutative.*

Corollary 2.3. *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Let δ be a nonzero derivation of R such that $\delta(x^*) \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. We are given that δ a nonzero derivation of R such that $\delta(x^*) \in Z(R)$ for all $x \in R$. For any $x \in R$, x^* also is an element of R. Substitution $[x, x^*]$ for x in the given assertion, we obtain $\delta([x, x^*]) \in Z(R)$ for all $x \in R$. Hence R is commutative by Theorem 2.2. This proves the corollary. □

Theorem 2.4. *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Let δ be a nonzero derivation of R. Then the following conditions are mutually equivalent:*

(*i*) *$\delta([x, x^*]) \in Z(R)$ for all $x \in R$;*

(*ii*) *$\delta(x \circ x^*) \in Z(R)$ for all $x \in R$;*

(*iii*) *$\delta(x^*) \in Z(R)$ for all $x \in R$;*

(*iv*) *R is commutative.*

Proof. We assume that (iv) holds (*i.e.*, $Z(R) = R$). Then for $x \in R$, $\delta(x)$ is also in $Z(R)$. Henceforth, we conclude that $\delta([x, x^*]) \in Z(R)$ for all $x \in R$, $\delta(x \circ x^*) \in Z(R)$ for all $x \in R$ and $\delta(x^*) \in Z(R)$ for all $x \in R$. Thus $(iv) \Rightarrow (i)$, $(iv) \Rightarrow (ii)$ and $(iv) \Rightarrow (iii)$. We need to prove that $(i) \Rightarrow (iv)$, $\Rightarrow (iv)$ and $(iii) \Rightarrow (iv)$ Now we suppose that any one (i) or (ii) or (iii) holds that is, $\delta([x, x^*]) \in Z(R)$ for all $x \in R$, or $\delta(x \circ x^*) \in Z(R)$ for all $x \in R$ or $\delta(x^*) \in Z(R)$ for all $x \in R$. Hence, result is follows by Theorems 2.2, 2.3 & Corollary 2.3. This finishes the proof of the theorem. □

Corollary 2.4. *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Let δ be a nonzero derivation of R. Then the following conditions are mutually equivalent:*

(*i*) *$\delta([x, y]) \in Z(R)$ for all $x, y \in R$;*

(*ii*) *$\delta(x \circ y) \in Z(R)$ for all $x, y \in R$;*

(*iii*) *$\delta(x) \in Z(R)$ for all $x \in R$;*

(*iv*) *R is commutative.*

Concluding Remark

We conclude the our paper with the following open questions.

Open Question 1. *Let R be a semiprime ring with involution $'*'$ of the second kind and with suitable torsion restrictions on R. Let δ be a nonzero derivation of R such that $\delta([x, x^*]) = 0$ (or $\in Z(R)$) for all $x \in R$. Is R commutative ?*

Open Question 2. *Let R be a semiprime ring with involution $'*'$ of the second kind and with suitable torsion restrictions on R. Let δ be a nonzero derivation of R such that $\delta(x \circ x^*) = 0$ (or $\in Z(R)$) for all $x \in R$. Is R commutative ?*

3 Conclusion

In the present paper we study some criteria to establish the commutativity of prime rings with involution via derivations. In particular, we solve some $*$-differential identities involving derivations, and we describe the structure of prime rings with involution. In addition, we present some open problems for future research.

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Competing Interests

Authors have declared that no competing interests exist.

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