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Bayesian the Kalman Type Recursive Formulae

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, the Kalman filter for a variance term of state space models is derived. First, it is assumed that the innovation term of state space model have a GARCH structure and the Kalman filter is derived. Then, it is assumed that the error term of observation equation is GARCH and the Kalman filtering is surveyed. Finally, considering an inverse gamma prior distribution for variance of observation equation again the Kalman filter is proposed. A numerical example is also given. Finally a conclusion section is presented.

Keywords: Bayesian method; posterior distribution; recursive Kalman filter.

1 Introduction

The Kalman filter, used for adaptive control of a dynamic system, is very important in financial applications. It may also be applied for filtering, prediction and smoothing of a financial time series. Indeed, it is a set of recursive relations for conditional expectation of $t\text{-}th$ state given observations $\{y_t\}_{t\geq 1}$, in a linear state space model. For non-linear models with non-normal errors, others filters such as particle filter are advised. For a comprehensive review about the Kalman filter and its extensions, see [1]. [2] proposed a probabilistic approach for derivation of the Kalman filter by Bayesian method. To describe more, following [2], let

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$$y_t = b_t Z_t + \zeta_t, t \geq 1$$

denote the observation equation at which ζ_t is a sequence of independent innovations with normal distribution $N(0, w_t^2)$. It is also independent of Z_t for each $t \geq 1$. It is assumed that b_t is nonzero sequence of real numbers. Here, we suppose that the state space equation is given by

$$Z_t = a_{t,t-1} Z_{t-1} + u_t$$

where $a_{t,t-1}$ is transition constants necessary for passing from step $t-1$ to t. An important example of above model is the Capital Asset Pricing Model (CAPM), where

$$r_t{}^* - r_f = \beta_t (R_t^m - r_f) + \zeta_t, t \geq 1$$

at which $y_t = r_t{}^* - r_f$ is the excess return of risk asset, $R_t^m - r_f$ is the excess return of market, r_f is the riskless return and $Z_t = \beta_t$ the market systematic measure. [2] considered a normal distribution for u_t. However, one of important time series which is used frequently in financial engineering is the GARCH model. It is a non-linear model and in the presence of a GARCH process, the the Kalman filter may not work. In this case, the particle filter is often advised. However, in this paper, using Kyriazis's method, the Kalman type recursive formulae for a state space model with GARCH (p,q) series as error process is proposed.

The Kalman filter is a mathematical power tool that is playing an increasingly important role in finance. It gives optimal recursive estimator of unknown parameters. Since it is in recursive form, new measurements can be processed once newcomer observations arrived. The problem of the Kalman filter is still valid and very import in many fields of science for example in financial applications. This fact is demonstrated by a series papers cited by this manuscript. Indeed, the Kalman filter is increasingly used in financial applications. A comprehensive review about the application of the Kalman filtering in financial models may be found in [3]. [4] studied the estimation of ARCH time series using adaptive filtering. [5] studied the Kalman filter for efficient uncertainty propagation. A comprehensive reference in the Kalman filtering is [6]. [7] studied high-dimensional prior and posterior in the Kalman filter variants. [8] studied the application of The The Kalman filter in hedge fund problems. [9] considered the Kalman filter for large-scale systems. [10] derived the fast the Kalman filtering on quasi-linear trees. [11] proposed R codes of the Kalman filters. [12] extracted the Kalman filtering and backward smoothing via a perturbative approach. Some extensions about the Kalman filter is found in [13]. [14] applied the Kalman filter to a fuzzy GARCH model. [15] applied the Kalman filter to stochastic volatility model. [16] introduced the concept of the stable robust Kalman filter. [17] derived the Kalman filtering with random coefficients and contractions.

This paper is organized as follows. In Section 2, the Kalman filter are derived under first GARCH modeling for error of state space and second GARCH modeling for error of observations. Finally, the Kyriazis's method is applied for variance term by considering an inverse gamma prior distribution. Again, the Kalman filters are derived. A numerical example is also given. Finally a conclusion section is presented.

2 The Kalman Filter Derivation

Here, extensions of the Kalman filter are derived.

2.1 GARCH in state equation

Assume that $u_t = \sigma_t \varepsilon_t$ is a GARCH (p,q) time series where ε_t's are *iid* standard normally distributed random variables and

$$\sigma_t^2 = c_0 + \sum_{i=1}^{p} c_i \sigma_{t-i}^2 + \sum_{j=1}^{q} d_j r_{t-j}^2$$

where a_0, c_i and d_j are positive constants such that $\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j < 1$. Let σ_t, ε_t, ζ_t and Z_{t-1} are mutually independent. Denote $\digamma_t$ is the sigma-field generated by $\sigma_s; 1 \le s \le t$, that is $\digamma_t = \sigma\{\sigma_s; 1 \le s \le t\}$.

Following [2], suppose that conditional posterior distribution (at time $t-1$) of Z_{t-1} given $\digamma_{t-1}$ is normal with mean $\tilde{Z}_{t-1}$ and variance $\tilde{v}_{t-1}^2$. This technique is learnt from [2] to obtain the Kalman filters using the Bayesian method. However, his method adapted for conditional cases, in this note. Therefore, the conditional prior (at time t) of Z_t given $\digamma_t$ is normal with mean $Z_t^{\,*}$ and variance v_t^{*2}, where

$$\begin{cases} Z_t^{\,*} = a_{t,t-1}\tilde{Z}_{t-1} \\ v_t^{*2} = a_{t,1-1}^2 \tilde{v}_{t-1}^2 + \sigma_t^2 \end{cases}$$

One can note that $Z_t = b_t^{-1} y_t + b_t^{-1}\zeta_t$. Then, the Z_t given r_t has normal distribution $N(b_t^{-1}y_t, b_t^{-2}w_t^2)$. Therefore, the likelihood function is

$$\frac{|b_t|}{\sqrt{2\pi} w_t} \exp\left\{\frac{-b_t^2}{2w_t^2}\left(Z_t - \frac{y_t}{b_t}\right)^2\right\}.$$

Define $\pi_t = \frac{b_t^2 v_t^{*2}}{b_t^2 v_t^{*2} + w_t^2}$. Using the Bayes theorem, it is seen that the conditional posterior of Z_t given $\digamma_t$ is normal with mean $\tilde{Z}_t$ and variance $\tilde{v}_t^2$, where

$$\begin{cases} \tilde{Z}_t = \pi_t y_t + (1-\pi_t) Z_t^{\,*} \\ \tilde{v}_t^2 = b_t^{-1} w_t^2 \pi_t. \end{cases}$$

The above equations provides the conditional Kalman filter estimates given $\digamma_t$. To calculate them, it is enough to generate some realizations for $\sigma_s; 1 \le s \le t$. Given parameters of GARCH process are known, this is an easy task. However, the marginal Kalman filter are derived by averaging of the conditional Kalman filters with respect to $\digamma_t$. By the conditional posterior distribution, we mean the posterior distribution given $\digamma_t$, that is the information up to time t. The conditional prior is also the prior distribution given $\digamma_t$.

Remark 1. The classical Kalman filter involves the estimation of Z_t with respect to the observed σ- algebra σ(y_s, s ≤ t). Here, First, the Kalman filters are derived with respect to observations and the σ-algebra of $\digamma_t$ and then the marginal Kalman filter (the regular Kalman filters) are derived by averaging of the conditional Kalman filter with respect to $\digamma_t$.

2.2 GARCH in observation equation

Here, we suppose that there exists a GARCH process for observation errors. That is,

$$\begin{cases} y_t = b_t Z_t + \zeta_t \\ Z_t = a_{t,t-1} Z_{t-1} + u_t \end{cases}$$

where $\zeta_t = \sigma_t \varepsilon_t$ and ε_t are *iid* random variables come from $N(0,1)$ distribution and σ_t^2 constitutes a GARCH(p,q) time series and u_t has $N(0, v^2)$ distribution. Here, σ_t, ε_t and u_t are mutually independent. Let Z_{t-1} has a conditional posterior normal distribution with mean μ_{t-1} and variance γ_{t-1}^2. It is not difficult to see that Z_t has also conditional posterior normal distribution with the following parameters

$$\begin{cases} \mu_t = \dfrac{a_{t,t-1}\mu_{t-1}\sigma_t^2 + \theta_t^2 y_t b_t^2}{\theta_t^2 b_t^2 + \sigma_t^2} \\ \\ \qquad \gamma_t^2 = \dfrac{\sigma_t^2\theta_t^2 b_t^2}{\theta_t^2 b_t^2 + \sigma_t^2} \end{cases}$$

where $\theta_t^2 = a_{t,1-1}^2\gamma_{t-1}^2 + v^2$. Let $k_t = \frac{\theta_t^2 b_t^2}{\theta_t^2 b_t^2+\sigma_t^2}$ be the Kalman gain, then

$$\begin{cases} \mu_t = a_{t,t-1}\mu_{t-1} + k_t(y_t - a_{t,t-1}\mu_{t-1}) \\ \qquad \gamma_t^2 = \theta_t^2 b_t^2(1-k_t). \end{cases}$$

The above filters are derived by conditioning on $\digamma_t$. Again, the marginal Kalman filter are derived by averaging of the conditional Kalman filters with respect to $\digamma_t$.

2.3 The Kalman filter for variance

Let $y_t = \sigma_t\varepsilon_t$ be the observation equation where ε_t has standard normal distribution. One of important example is the first order autoregressive GARCH model for the return of risky asset, where this model is frequently used in financial risk management, that is

$$y_t = r_t - ar_{t-1} = \sigma_t\varepsilon_t$$

Let $\gamma_t = \frac{1}{\sigma_t^2}$ and suppose that γ_{t-1} has gamma distribution with parameters α_{t-1} and β_{t-1}. Let $\sigma_t = b_{t,t-1}\sigma_{t-1}$, be the state equation where $b_{t,t-1}$ is a sequence of positive real numbers. Again, it is seen that the updated prior of γ_t is gamma distribution with parameters α_{t-1} and $\frac{\beta_{t-1}}{b_{t,t-1}^2}$. Using the likelihood function given by $\frac{1}{\sqrt{\sigma_t^2}}\exp\left(\frac{-y_t^2}{2\sigma_t^2}\right)$. The update posterior of γ_t is gamma distribution with parameters

$$\begin{cases} \alpha_t = \alpha_{t-1} + 0.5 \\ \dfrac{1}{\beta_t} = \dfrac{b_{t,t-1}^2}{\beta_{t-1}} + y_t^2. \end{cases}$$

Therefore, $\alpha_t = \alpha_0 + 0.5t$ and $\alpha_t \to \infty$ as $t \to \infty$ .The mean and variance of β_t are

$$E\left(\frac{1}{\beta_t}\right) = b_{t,t-1}^2 E\left(\frac{1}{\beta_{t-1}}\right) + \sigma_t^2$$

$$var\left(\frac{1}{\beta_t}\right) = b_{t,t-1}^4 var\left(\frac{1}{\beta_{t-1}}\right) + 3\sigma_t^4$$

Given β_{t-1}, variable $\frac{1}{\beta_t}$ has location-scale distribution of chi-square distribution with one degree of freedom at which the location and scale parameters are $\frac{b_{t,t-1}^2}{\beta_{t-1}}$ and σ_t^2, respectively. As follows, the mean and variance of σ_t^2 are studied. It is easy to see that

$$\begin{cases} \theta_t = E(\sigma_t^2) = E(\gamma_t^{-1}) = \dfrac{1}{\beta_t(\alpha_t - 1)} \\ \\ \qquad v_t^2 = \dfrac{1}{\beta_t^2(\alpha_t - 1)(\alpha_t - 2)} \end{cases}$$

Then, $v_t^2 = (\alpha_t - 2)\theta_t^2$. Therefore,

$$\theta_t = \frac{1}{(\alpha_{t-1} - 1)}\left(\frac{b_{t,t-1}^2}{\beta_{t-1}} + y_t^2\right)\frac{(\alpha_{t-1} - 1)}{(\alpha_{t-1} - 0.5)}.$$

For large t, $\frac{(\alpha_{t-1}-1)}{(\alpha_{t-1}-0.5)} = O(1)$ then

$$\theta_t = \frac{b_{t,t-1}^2}{\beta_{t-1}(\alpha_{t-1} - 1)} + \frac{y_t^2}{\alpha_{t-1} - 1} = b_{t,t-1}^2\theta_{t-1} + \frac{y_t^2}{\alpha_{t-1} - 1}.$$

That is

$$\theta_t = b_{t,t-1}^2\theta_{t-1} + \frac{y_t^2}{\alpha_{t-1} - 1}$$

It is also easy to see that

$$v_t^2 = b_{t,t-1}^2 v_{t-1}^2 + \frac{y_t^2}{(\alpha_{t-1} - 1)(\alpha_{t-1} - 2)}.$$

The following proposition summarizes the above discussion.

Proposition 1. Suppose that $\sigma_t^2 = O(t^c)$ where $0 < c < 1$, then

$$\begin{cases} \theta_t = b_{t,t-1}^2\theta_{t-1} \\ \\ v_t^2 = b_{t,t-1}^2 v_{t-1}^2 \end{cases}$$

The Kalman filtering uses the normality assumption for ε_t. However, it is not a realistic assumption, in practice. Historical data analysis shows that fat tail distributions are usually suitable for ε_t. Thus, in the case of heavy tail distribution, The Kalman filter fails and some extensions like the particle filters or generally the Bayes filter are needed. Using the Chapman-Kolmogorov equation, the Bayes prediction step is given by

$$f(\beta_t|y_{t-1}, \dots, y_1) = \int f(y_t|\beta_{t-1})\, f(\beta_{t-1}|y_{t-1}, \dots, y_1)\, d\beta_{t-1},$$

and the Bayes update equation is

$$f(\beta_t|y_t, \dots, y_1) \propto f(y_t|\beta_t) f(\beta_t|y_{t-1}, \dots, y_1).$$

In order to initialize the recurrence algorithm, it is assumed that the initial return R_0 has known probability distribution $f(y_0)$. Using the Bayes filter, the probability distribution $f(\beta_t|y_t, \dots, y_1)$ and $f(\beta_t|y_{t-1}, \dots, y_1)$ are not computed. Only, the expectations $E(\beta_t|y_t, \dots, y_1)$ and $E(\beta_t|y_{t-1}, \dots, y_1)$ are calculated.

2.4 A numerical example

Here, a numerical example is given. First, a special case is studied as a remark.

Remark 2. As special case, let $b_t = \sigma_t = B$ and $a_{t,t-1} = A$. Then, $k_t = \frac{\theta_t^2}{1+\theta_t^2}$ and $\gamma_t^2 = B^2 k_t$. Also, $\theta_t^2 = A^2\gamma_{t-1}^2 + v^2$. Then, $\mu_t = A\mu_{t-1} + k_t(y_t - A\mu_{t-1})$.

Then, for example let, $= B = 0.1, v = 0.2$. Here, it is assumed that y_t comes from a standard normal distribution and the above formula is applied to this case. The time series plot of Kalman estimates are plotted in the following figure which are oscillated around zero.

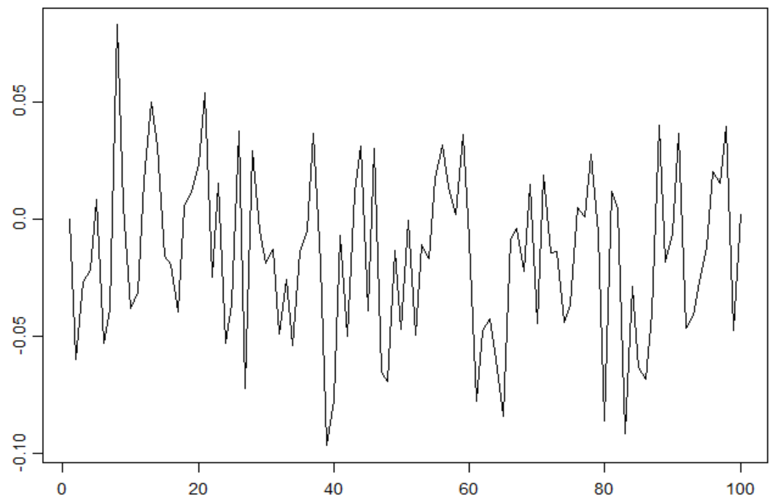

Fig. 1. The Kalman estimates of mu

3 Conclusions

In a state space model, the Kalman filter for a variance is derived. The first assumption was that the innovation term of state space model had a GARCH structure. Then, it was assumed that the error term of observation equation is GARCH. In both cases, the Kalman filtering is surveyed. Finally, considering an inverse gamma prior distribution for variance of observation equation again the Kalman filter is proposed.

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Competing Interests

Author has declared that no competing interests exist.

References

[1] Luenberger DC, Luenberger DG. Introduction to dynamic systems: Theory, models, and application. John Wiley. USA; 1979.

[2] Kyriazis G. A simplified derivation of scalar Kalman filter using Bayesian probability theory. *6th* International Workshop on Analysis of Dynamic Measurements. Goteborg. Sweden; 2011.

[3] Harvey AC. Forecasting, structural time series models and the Kalman filter. Cambridge University Press. Cambridge; 1989.

[4] Nelson DB, Foster DP. Asymptotic filtering theory for univariate ARCH models. Econometrica. 1994;62(1):1-41.

[5] Treebushny D, Madsen H. On the construction of a reduced rank square-root Kalman filter for efficient uncertainty propagation. Future Generation Computer Systems. 2005;21(1):1047–1055.

[6] Simon D. Optimal state estimation: The Kalman, h-infinity, and nonlinear approaches. Wiley. USA; 2006.

[7] Furrer R, Bengtsson T. Estimation of high-dimensional prior and posterior covariance matrices in the Kalman filter variants. Journal of Multivariate Analysis. 2007;98(1):227–255.

[8] Racicot F, Theoret R. A study of dynamic market strategies of hedge funds using the Kalman filter. Journal of Wealth Management. 2007;10(1):94-106.

[9] Khan UA, Moura JMF. Distributing Kalman filter for large-scale systems. IEEE Transactions on Signal Processing. 2008;56(1):4919–4935.

[10] Paninski L. Fast Kalman filtering on quasilinear dendritic trees. Journal of Computational Neuroscience. 2010;28(1):211–228.

[11] Tusell F. The Kalman filtering in R. Journal of Statistical Software. 2011;39:1-27.

[12] Pnevmatikakis EA, Rahnama-Rad K, Huggins J, Paninski L. Fast Kalman filtering and forward–backward smoothing via a low-rank perturbative approach. Journal of Computational and Graphical Statistics. 2014;23(1):316-339.

[13] Nunez C. The Kalman-Bucy filter revisited: Discrete and continuous dynamical systems. Discrete and Continuous Dynamical System. 2014;34(1):4139-4153.

[14] Hung JC. Robust Kalman filter based on a fuzzy GARCH model to forecast volatility using particle swarm optimization. Soft Computing. 2015;19(1):2861–2869.

[15] Ghosh H, Gurung B. The Kalman filter-based modeling and forecasting of stochastic volatility with threshold. Journal of Applied Statistics. 2015;42:492-507.

[16] Kuok S, Yuen K. Stable robust extended Kalman filter. Journal of Aerospace Engineering. 2016;10 (1):61-75.

[17] Bougerol P. The Kalman filtering with random coefficients and contractions. SIAM Journal on Control and Optimization. 2016;31(1):942–959.

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