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Unified Convergence Criteria of Derivative-Free Iterative Methods for Solving Nonlinear Equations

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Abstract: A local and semi-local convergence is developed of a class of iterative methods without derivatives for solving nonlinear Banach space valued operator equations under the classical Lipschitz conditions for first-order divided differences. Special cases of this method are well-known iterative algorithms, in particular, the Secant, Kurchatov, and Steffensen methods as well as the Newton method. For the semi-local convergence analysis, we use a technique of recurrent functions and majorizing scalar sequences. First, the convergence of the scalar sequence is proved and its limit is determined. It is then shown that the sequence obtained by the proposed method is bounded by this scalar sequence. In the local convergence analysis, a computable radius of convergence is determined. Finally, the results of the numerical experiments are given that confirm obtained theoretical estimates.

Keywords: iterative method; Banach space; divided difference; semi-local convergence; local convergence; error analysis; sufficient convergence conditions

MSC: 49M15; 47H17; 65J15; 65G99; 41A25

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1. Introduction

One of the greatest challenges numerical functional analysis and other computational disciplines the task of approximating a locally unique solution x_* of the nonlinear equation

$$F(x) = 0, \tag{1}$$

for $F : \Omega \subseteq X \to X$, F is a continuous operator, acting between Banach space X and itself. The solution x_* is needed in closed or analytical form but this is possible only in special cases. That is why iterative solution methods are used to generate a sequence approximating x_* provided certain conditions are verified on the initial information.

Newton's method (NM) defined for each $n = 0, 1, 2, \ldots$

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{2}$$

has been used extensively to generate such a sequence converting quadratically to x_* [1,2].

However, there are some difficulties with the implementation of it in case the inverse of linear operator $F'(x_n)$ is very expensive to calculate or even does not exist.

This difficulty is handled by considering iterative methods of the form

$$x_{n+1} = x_n - T_n^{-1}F(x_n) \; for\ each\ n = 0, 1, 2, \ldots, \tag{3}$$

where $T_n = [G_n, H_n; F]$, $[\cdot,\cdot;F] : \Omega \times \Omega \to L(X, X)$, $G_n = G(x_n, x_{n-1}) = ax_n + bx_{n-1} + cF(x_n)$, $H_n = H(x_n, x_{n-1}) = dx_n + px_{n-1} + qF(x_n)$, and $a,\ b,\ c,\ d,\ p$ an q are real numbers.

Motivation for writing this article. Some popular methods are special cases of (3):
Newton: set $a = d = 1$, $b = c = p = q = 0$ provided F is Fréchet-differentiable;
Secant [1,3,4]: set $a = p = 1$, $b = c = d = q = 0$;
Kurchatov [5–8]: pick $a = 2$, $b = -1$, $p = 1$, $c = d = q = 0$;
Steffensen [1,9]: pick $a = d = 1$, $c = 1$ and $b = p = q = 0$.

The convergence order of these iterative methods is 2, 1.6..., 2 and 2, respectively, [1,2,7,9]. However, the convergence ctiteria differ, rendering the comparison between them difficult [10–12].

Other choices of the parameters lead to less well-known methods or new methods [1,7,8,13]. Iterative methods are constructed usually based on geometrical or algebraic considerations. Ours is the latter. The introduction of these parameters and function evaluations allow for a greater flexibility, tighter error accuracy, and the handling of equations not possible before (see also numerical section). The choice $qF(x_n)$ is not necessary more appropriate.

Semi-local and local constitute two types of convergence for iterative methods.

In the semi-local convergence analysis, information is used from the initial point x_0 to find usually sufficient convergence criteria for the method (3). A priori estimates on the norms $\|x_n - x^*\|$ are also obtained. In the local convergence analysis, data about the solution x^* is taken into account to determine the radius of convergence for the method (3). Moreover, usually upper error bounds are calculated for the norms $\|x_n - x^*\|$. Generalized Lipschitz-type conditions are used for both types of convergence.

The novelty of the article. Therefore, it is important to study the convergence of method (3) in both the semi-local (Sections 2 and 3) as well as the local convergence (Section 4) case. Our technique allows for a comparison between the convergence criteria of these methods. The new convergence criteria can be weaker than those ones given if the methods are studied separately. Section 5 contains the numerical examples, and Section 6 contains the conclusions.

2. Majorizing Sequence

It is convenient for the semi-local convergence analysis of method (3) to introduce some parameters, sequences, and functions. Let L_0, L, λ, h, η_0, η, and $\bar{\eta}$ be given parameters. Define the parameters

$$A = |a| + |d| + \lambda|c|, \quad B = |b| + |p|,$$

$$C = (|c| + |p|)h, \quad \alpha = |1-a| + |1-d|,$$

$$\beta = \lambda(|c| + |q|), \quad \gamma = |1-a-b| + |1-d-p|,$$

$$\delta = |1-a-b|\bar{\eta} + |c|\eta_0 + |1-d-p|\bar{\eta} + |q|\eta_0,$$

$$t_{-1} = 0, \quad t_0 = h, \quad t_1 = \eta + h,$$

sequences

$$\mu_{n+1} = L_0(A(t_{n+1} - t_0) + B(t_n - t_0) + C),$$

$$\lambda_{n+1} = L(t_{n+1} - t_n + \alpha(t_n - t_{n-1}) + \beta(t_n - t_0) + \gamma(t_{n+1} - t_0) + \delta),$$

$$t_{n+2} = t_{n+1} + \frac{\lambda_{n+1}}{1 - \mu_{n+1}}(t_{n+1} - t_n). \tag{4}$$

We shall show that $\{t_n\}$ is a majorizing sequence for $\{x_n\}$ under certain conditions. Moreover, define parameters θ_i, $i = 1, 2, \ldots, 8$ by

$$\theta_1 = \frac{L_0 A}{1 - L_0 C}, \quad \theta_2 = \frac{L_0 B}{1 - L_0 C}, \quad \theta_3 = \frac{L}{1 - L_0 C}, \quad \theta_4 = \frac{L\alpha}{1 - L_0 C},$$

$$\theta_5 = \frac{L\beta}{1 - L_0 C}, \quad \theta_6 = \frac{L\gamma}{1 - L_0 C}, \quad \theta_7 = \frac{L\delta}{1 - L_0 C}, \quad \theta_8 = (\theta_5 + \theta_6)h + \theta_7,$$

$$s_{-1} = 0,\ s_0 = h,\ s_1 = \eta + h,$$

and sequences

$$m_{n+1} = \theta_1(t_{n+1} - t_0) + \theta_2(t_n - t_0),$$

$$l_{n+1} = \theta_3(t_{n+1} - t_n) + \theta_4(t_n - t_{n-1}) + \theta_5(t_n - t_0) + \theta_6(t_{n-1} - t_0) + \theta_7,$$

$$s_{n+2} = s_{n+1} + \frac{l_{n+1}(s_{n+1} - s_n)}{1 - m_{n+1}}. \tag{5}$$

We shall study the simplified version $\{s_n\}$ of sequence $\{t_n\}$.

Furthermore, define the interval $[0, 1)$ quadratic polynomial

$$g(t) = (\theta_1 + \theta_3)t^2 + (\theta_2 + \theta_4 - \theta_3 + \theta_5)t + \theta_6 - \theta_4,$$

function

$$Q_\infty(t) = \frac{\theta_5\eta}{1-t} + \frac{\theta_6\eta}{1-t} + \frac{\theta_1\eta t}{1-t} + \frac{\theta_2\eta t}{1-t} + \theta_1 ht + \theta_1 ht - t + \theta_8$$

and sequence

$$\begin{aligned} Q_n(t) &= \theta_3\eta t^n + \theta_4\eta t^{n-1} + \theta_5\left(h + \frac{1-t^n}{1-t}\eta\right) + \theta_6\left(h + \frac{1-t^{n-1}}{1-t}\eta\right) + \theta_7 \\ &\quad + t\theta_1\left(h + \frac{1-t^n}{1-t}\eta\right) + t\theta_2\left(h + \frac{1-t^{n-1}}{1-t}\eta\right) - t + \theta_8 \\ &= \theta_3\eta t^n + \theta_4\eta t^{n-1} + \theta_5(1 + t + \ldots + t^{n-1})\eta + \theta_6(1 + t + \ldots + t^{n-2})\eta \\ &\quad + \theta_7 + t\theta_1(1 + t + \ldots + t^{n-1})\eta + t\theta_2(1 + t + \ldots + t^{n-2})\eta \\ &\quad + t\theta_1 h + t\theta_2 h - t + \theta_8. \end{aligned}$$

Suppose that either of the following conditions hold:

(I)

$$L_0 C < 1$$

equation $Q_\infty(t) = 0$ has a minimal solution $w \in (0, 1)$ satisfying

$$0 \le \frac{l_1}{1 - m_1} \le w$$

and

$$g(w) \ge 0.$$

(II)

$$L_0 C < 1$$

and w exists satisfying

$$0 \le \frac{l_1}{1 - m_1} \le w,$$

$$Q_1(w) \le 0$$

and

$$g(w) \ge 0.$$

Then, we can show the following result on majorizing sequences for method (3).

Lemma 1. *Under conditions (I) or (II), sequence $\{s_n\}$ generated by (5) is nondecreasing, bounded from above by $s_{**} = h + \dfrac{\eta}{1-w}$ and converges to its unique least upper bound $s_* \in [h + \eta, s_{**}]$.*

Proof. Induction is used to show

$$0 \leq \frac{l_{k+1}}{1-m_{k+1}} \leq w \tag{6}$$

and

$$m_{k+1} < 1. \tag{7}$$

These estimates are true for $k=0$ by (I) (or II). It then follows from (5) that

$$0 \leq s_2 - s_1 \leq w(s_1 - s_0) = w\eta$$

and

$$s_2 \leq h + (1+w)\eta = h + \frac{1-w^2}{1-w}\eta \leq s_{**}.$$

Assume

$$0 \leq s_{k+1} - s_k \leq w^k \eta. \tag{8}$$

Then, we also have

$$\begin{aligned} s_{k+1} &\leq s_k + w^k\eta \leq s_{k-1} + w^{k-1}\eta + w^k\eta \leq \ldots \\ &\leq s_1 + \eta + w\eta + \ldots + w^k\eta = h + \frac{1-w^{k+1}}{1-w}\eta \leq s_{**}. \end{aligned} \tag{9}$$

Evidently, if we use (5), (8), and (9) estimates (6) and (7) are true if

$$\begin{aligned} &\theta_3\eta w^k + \theta_4\eta w^{k-1} + \theta_5\left(h + \frac{1-w^k}{1-w}\eta\right) + \theta_6\left(h + \frac{1-w^{k-1}}{1-w}\eta\right) + \theta_7 \\ &+t\theta_1\left(h + \frac{1-w^k}{1-w}\eta\right) + w\theta_2\left(h + \frac{1-w^{n-1}}{1-w}\eta\right) - w \leq 0. \end{aligned} \tag{10}$$

Define recurrent functions Q_k on the interval $[0,1)$ by

$$\begin{aligned} Q_k(t) = {} & \theta_3\eta t^k + \theta_4\eta t^{k-1} + \theta_5\eta(1+t+\ldots+t^{k-1}) + \theta_6\eta(1+t+\ldots+t^{k-2}) \\ &+t\eta\theta_1(1+t+\ldots+t^{k-2}) + t\eta\theta_2(1+t+\ldots+t^{k-2}) \\ &+t\theta_1 h + t\theta_2 h - t + \theta_8. \end{aligned} \tag{11}$$

Then, we can show instead of (10) that

$$Q_k(w) \leq 0. \tag{12}$$

Next, we relate two consecutive functions Q_k. By the definition of these functions we have

$$Q_{k+1}(t) = Q_{k+1}(t) - Q_k(t) + Q_k(t) = Q_k(t) + g(t)t^{k-1}\eta. \tag{13}$$

Case I. We have by (13) $Q_{k+1}(w) = Q_k(w)$ since $g(w) = 0$. Define function Q_∞ by

$$Q_\infty(t) = \lim_{k \to \infty} Q_k(t). \tag{14}$$

Then, we have by (11) and (14) that

$$Q_\infty(t) = \frac{\theta_5\eta}{1-t} + \frac{\theta_6\eta}{1-t} + \frac{\theta_1\eta t}{1-t} + \frac{\theta_2\eta t}{1-t} + \theta_1 ht + \theta_2 ht - t + \theta_8. \tag{15}$$

It follows by $Q_\infty(w) = Q_\infty(w)$, (12) and (15) that we can show instead that

$$Q_\infty(w) \leq 0, \tag{16}$$

which is true by the choice of w.

Case II. By $g(w) \leq 0$ and (13), we have

$$Q_{k+1}(w) \leq Q_k(w). \tag{17}$$

Thus, we can show instead of (12) that

$$Q_1(w) \leq 0.$$

which is true by the definition of w. The induction for (6) and (7) is completed. Hence, in either case (I) or (II) sequence $\{s_n\}$ is nondecreasing and bounded from above by s_{**} and as such it converges to its unique least upper bound s_*. □

Remark 1. *(a) Clearly sequence $\{t_n\}$ can replace $\{s_n\}$ in Lemma 1 (since they are equivalent).*

(b) It follows from the proof on the Theorem 1 that the convergence of the method (3) depends on the majorizing sequence (4). Sufficient convergence criteria for the majorizing sequence are given in Lemma 1.

Next, more general sufficient convergence criteria are developed so that the conditions of the Lemma 1 imply those of the Lemma 2 but not necessarily vice versa.

Lemma 2. *Suppose that there exists $\rho > 0$ such that for each $n = 0, 1, 2, \ldots$*

$$\mu_{n+1} < 1 \quad \text{and} \quad t_n < \rho. \tag{18}$$

Then, the following assertion holds

$$0 \leq t_n \leq t_{n+1} < \rho \quad \text{and} \quad t_n < \rho. \tag{19}$$

and $\rho_* \in [0, \rho]$ exists such that

$$\lim_{n \to \infty} t_n = \rho_*. \tag{20}$$

Proof. The definition of the sequence $\{t_n\}$ given by the formula (4) and the condition (18) imply the assertion (19) from which the item (20) is implied. □

Remark 2. *A possibly choice for ρ under the conditions of the Lemma 1 is s_*.*

3. Semi-Local Convergence

The following condition (R) shall be used in the semi-local convergence.

(R_1) $x_{-1}, x_0 \in \Omega$, $h \geq 0$, $\eta \geq 0$, $\eta_0 \geq 0$, and $\bar{\eta} \geq 0$ exist such that

$$\|x_{-1} - x_0\| \leq h, \quad \|T_0^{-1}F(x_0)\| \leq \eta, \quad \|F(x_0)\| \leq \eta_0 \text{ and } \|x_0\| \leq \bar{\eta}.$$

(R_2) $L_0 \geq 0$, $L \geq 0$, and $\lambda \geq 0$ exist such that for all $x, y, z \in \Omega$

$$\|T_0^{-1}([G(y,x), H(y,x); F] - T_0)\| \leq L_0(\|G(y,x) - G_0\| + \|H(y,x) - H_0\|),$$

$$\|T_0^{-1}([z, y; F] - [G(y,x), H(y,x); F])\| \leq L(\|z - G(y,x)\| + \|y - H(y,x)\|)$$

and

$$\|F(y) - F(x_0)\| \leq \lambda\|y - x_0\|.$$

(R_3) Conditions of Lemma 1 hold with s_* also satisfying

$$\|(a + b - 1)x_0 + cF(x_0)\| \leq s_*(1 - |a| - |b| - \lambda|c|)$$

and
$$\|(d+p-1)x_0 + qF(x_0)\| \leq s_*(1-|d|-|p|-\lambda|q|).$$

(R_4) $U[x_0, s_*] \subset \Omega$.

Next, we show the semi-local convergence analysis of method (3) using conditions (R) and the preceding notation.

Theorem 1. *Suppose that conditions (R) hold. Then, sequence $\{x_n\}$ starting with x_{-1}, $x_0 \in U[x_0, s_*]$ and generated by method (3) is well-defined in $U[x_0, s_*]$, remains in $U[x_0, s_*]$, and converges to a solution $x_* \in U[x_0, s_*]$ of equation $F(x) = 0$.*

Proof. We shall show that $\{t_k\}$ is a majorizing sequence for $\{x_k\}$ using induction. Notice that $\|x_0 - x_{-1}\| \leq t_0 - t_{-1}$ and $\|x_1 - x_0\| \leq t_1 - t_0$. Suppose $\|x_{k+1} - x_k\| \leq t_{k+1} - t_k$.

First, we show that linear operator $T_{k+1}^{-1} \in L(X, X)$ exists. We have by the first condition in (R_2) that
$$\begin{aligned}\|T_0^{-1}(T_{k+1} - T_0)\| &= \|T_0^{-1}([G_{k+1}, H_{k+1}; F] - T_0)\| \\ &\leq L_0(\|G_{k+1} - G_0\| + \|H_{k+1} - H_0\|). \end{aligned} \tag{21}$$

However, we have by (R_2) and (R_3)
$$\|ax_{k+1} + bx_k + cF(x_{k+1}) - x_0\| \leq \|a(x_{k+1} - x_0) + b(x_k - x_0) + c(F(x_{k+1}) - F(x_0))$$
$$+ax_0 + bx_0 + cF(x_0) - x_0\| \leq \|(a+b-1)x_0 + cF(x_0)\| + |a|s_* + |b|s_* + |c|\lambda s_* \leq s_*,$$
and similarly
$$\|dx_{k+1} + px_k + qF(x_{k+1}) - x_0\| \leq \|(d+p-1)x_0 + qF(x_0)\|$$
$$+|d|s_* + |p|s_* + |q|\lambda s_* \leq s_*,$$
thus, the iteration $ax_{k+1} + bx_k + cF(x_{k+1})$, $dx_{k+1} + px_k + qF(x_{k+1})$ belong in $U[x_0, s_*]$.

Moreover, we have
$$\begin{aligned}\|G_{k+1} - G_0\| &= \|ax_{k+1} + bx_k + cF(x_{k+1}) - ax_0 + bx_{-1} + cF(x_0)\| \\ &\leq \|a(x_{k+1} - x_0) + b(x_k - x_0) + c(F(x_{k+1}) - F(x_0))\| \\ &\leq |a|\|x_{k+1} - x_0\| + |b|\|x_k - x_0 + x_0 - x_{-1}\| \\ &\quad +|c|\|F(x_{k+1}) - F(x_0)\| \\ &\leq |a|(t_{k+1} - t_0) + |b|(t_k - t_0) + |b|h + |c|\lambda(t_{k+1} - t_0). \end{aligned} \tag{22}$$

Similarly, it follows
$$\|H_{k+1} - H_0\| \leq |d|(t_{k+1} - t_0) + |p|(t_k - t_0) + |p|h + |q|\lambda(t_{k+1} - t_0), \tag{23}$$

hence (21) gives by summing up
$$\|T_0^{-1}(T_{k+1} - T_0)\| \leq \mu_{k+1} < 1$$

by Lemma 1, so T_{k+1}^{-1} exists and
$$\|T_{k+1}^{-1} T_0\| \leq \frac{1}{1 - \mu_{k+1}}. \tag{24}$$

Furthermore, we can write

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - T_k(x_{k+1} - x_k) \\ &= ([x_{k+1}, x_k; F] - T_k)(x_{k+1} - x_k) \\ &= ([x_{k+1}, x_k; F] - [G_k, H_k; F])(x_{k+1} - x_k). \end{aligned} \tag{25}$$

Using (R_2) and (25), we obtain

$$\|T_0^{-1}F(x_{k+1})\| \leq L(\|x_{k+1} - G_k\| + \|x_k - H_k\|)\|x_{k+1} - x_k\|. \tag{26}$$

However, we also have

$$\begin{aligned} x_{k+1} - G_k &= x_{k+1} - ax_k - bx_{k-1} - cF(x_k) = x_{k+1} - x_k \\ &\quad +(1-a)(x_k - x_{k-1}) + (1-a)x_{k-1} - bx_{k-1} - cF(x_k) \\ &= x_{k+1} - x_k + (1-a)(x_k - x_{k-1}) + (1-a-b)(x_{k-1} - x_0) \\ &\quad +(1-a-b)x_0 - c(F(x_k) - F(x_0)) - cF(x_0), \end{aligned}$$

thus

$$\begin{aligned} \|x_{k+1} - G_k\| &\leq \|x_{k+1} - x_k\| + |1-a|\|x_k - x_{k-1}\| + |1-a-b|\|x_{k-1} - x_0\| \\ &\quad +|1-a-b|\|x_0\| + \lambda|c|\|x_k - x_0\| + |c|\|F(x_0)\| \\ &\leq t_{k+1} - t_k + |1-a|(t_k - t_{k-1}) + |1-a-b|(t_{k+1} - t_0) \\ &\quad +|1-a-b|\bar{\eta} + \lambda|c|(t_k - t_0) + |c|\eta_0, \end{aligned}$$

Similarly,

$$\begin{aligned} x_k - H_k &= x_k - dx_k - px_{k-1} - qF(x_k) \\ &= (1-d)(x_k - x_{k-1}) + (1-d-p)(x_{k-1} - x_0) \\ &\quad +(1-d-p)x_0 - qF(x_k), \end{aligned}$$

so

$$\begin{aligned} \|x_k - H_k\| &\leq |1-d|(t_k - t_{k-1}) + |1-d-p|(t_{k-1} - t_0) + |1-d-p|\bar{\eta} \\ &\quad +|q|\lambda(t_k - t_0) + |q|\eta_0, \end{aligned}$$

hence,

$$\begin{gathered} \|x_{k+1} - G_k\| + \|x_k - H_k\| \leq t_{k+1} - t_k + |1-a|(t_k - t_{k-1}) \\ +|1-a-b|(t_{k+1} - t_0) + |1-a-b|\bar{\eta} + \lambda|c|(t_k - t_0) + |c|\eta_0 \\ +|1-d|(t_k - t_{k-1}) + |1-d-p|(t_{k-1} - t_0) + |1-d-p|\bar{\eta} \\ +|q|\lambda(t_k - t_0) + |q|\eta_0 \\ \leq t_{k+1} - t_k + \alpha(t_k - t_{k-1}) + \beta(t_k - t_0) + \gamma(t_{k-1} - t_0) + \delta. \end{gathered} \tag{27}$$

Therefore, by (26), (27) and the definition of sequence λ_{k+1}, we obtain

$$\|T_0^{-1}F(x_{k+1})\| \leq \lambda_{k+1}(t_{k+1} - t_k). \tag{28}$$

It then follows from (3), (24), and (28) that

$$\|x_{k+2} - x_{k+1}\| \leq \|T_{k+1}^{-1}T_0\|\|T_0^{-1}F(x_{k+1})\| \leq \frac{\lambda_{k+1}(t_{k+1} - t_k)}{1 - \mu_{k+1}} = t_{k+2} - t_{k+1}$$

and

$$\begin{aligned}\|x_{k+2}-x_0\| &\leq \|x_{k+2}-x_{k+1}\|+\|x_{k+1}-x_k\|+\ldots+\|x_1-x_0\| \\ &\leq t_{k+2}-t_0 \leq t_{k+2} \leq t_{**}.\end{aligned}$$

It follows that sequence $\{x_k\}$ is Cauchy (since $\{t_k\}$ is Cauchy as convergence by Lemma 4) and as such it converges to some $x_* \in U[x_0, s_*]$. By letting $k \to \infty$ in (28), we conclude $F(x_*) = 0$. $\square$

Remark 3. *Clearly, the conditions of Lemma 2 and ρ can replace Lemma 1 and s_* in Theorem 1.*

4. Local Convergence

Suppose:

(C_1) There exists a simple solution $x_* \in \Omega$ of equation $F(x) = 0$.

(C_2) For each $x, y \in \Omega$

$$\|F'(x_*)^{-1}([G(y,x),H(y,x);F]-F'(x_*))\| \leq l_0(\|G(y,x)-x_*\|+\|H(y,x)-x_*\|),$$

$$\|F'(x_*)^{-1}([G(y,x),H(y,x);F]-[y,x_*;F])\| \leq l(\|G(y,x)-y\|+\|H(y,x)-x_*\|),$$

$$\|F(y)\| \leq \lambda\|y-x_*\|.$$

(C_3) The parameter r_* satisfies the conditions

$$\|(a+b-1)x_*\| \leq r_*(1-|a|-|b|-\lambda|c|)$$

and

$$\|(d+p-1)x_*\| \leq r_*(1-|d|-|p|-\lambda|q|).$$

(C_4) $U(x_*, r_*) \subset \Omega$, where $r_* = \dfrac{1}{2l_0+3l}$.

Theorem 2. *Suppose that conditions (C) hold. Then, sequence $\{x_n\}$ starting with x_{-1}, $x_0 \in U(x_*, r_*)$ and generated by method (3) is well-defined in $U(x_*, r_*)$, remains in $U(x_*, r_*)$ and converges to a solution x_*.*

Proof. We have by (C_2) and (C_3) that

$$\|ax_k+bx_{k-1}+cF(x_k)-x_*\| \leq \|a(x_k-x_*)+b(x_{k-1}-x_*)+cF(x_k)$$

$$+ax_*+bx_*-x_*\| \leq \|(a+b-1)x_*\|+|a|r_*+|b|r_*+|c|\lambda r_* \leq r_*,$$

$$\|dx_k+px_{k-1}+qF(x_k)-x_*\| \leq \|(d+p-1)x_*\|+|d|r_*+|p|r_*+|q|\lambda r_* \leq r_*,$$

$$\|ax_k+bx_{k-1}+cF(x_k)-x_k\| \leq \|ax_k+bx_{k-1}+cF(x_k)-x_*\|+\|x_*-x_k\| \leq 2r_*,$$

$$\|F'(x_*)^{-1}(T_k-F'(x_*))\| \leq l_0(\|G_k-x_*\|+\|H_k-x_*\|) \leq 2l_0r_* < 1,$$

so

$$\|T_k^{-1}F'(x_*)\| \leq \frac{1}{1-l_0(\|G_k-x_*\|+\|H_k-x_*\|)}.$$

We also get by (C_2)

$$\|F'(x_*)^{-1}(T_k-[x_k,x_*;F])\| \leq l(\|H_k-x_k\|+\|G_k-x_*\|),$$

thus

$$\begin{aligned}\|x_{k+1}-x_*\| &= \|x_k - x_* - T_k^{-1}F(x_k)\| \\ &\le \|T_k^{-1}F'(x_*)\|\|F'(x_*)^{-1}(T_k-[x_k,x_*;F])(x_k-x_*)\| \\ &\le \|T_k^{-1}F'(x_*)\|\|F'(x_*)^{-1}(T_k-[x_k,x_*;F])\|\|x_k-x_*\| \\ &\le \frac{l(\|H_k-x_k\|+\|G_k-x_*\|)}{1-l_0(\|G_k-x_*\|+\|H_k-x_*\|)} < \|x_k-x_*\| < r_*,\end{aligned}$$

hence, the iterate $x_{k+1} \in U(x_*, r_*)$ and $\lim_{k\to\infty} x_k = x_*$. □

A uniqueness of the solution domain can be specified.

Proposition 1. *Suppose that there exists a solution $x_* \in \Omega$ of the equation $F(x) = 0$ such that for each $x \in U(x_*, \rho_1)$*

$$\|F'(x_*)^{-1}([x_*,x;F]-F'(x_*))\| \le l_1\|x-x_*\| \quad for \quad some \quad \rho_1, l_1 > 0; \tag{29}$$

$$l_1\rho_1 < 1. \tag{30}$$

Then, the point x_* is the only solution of the equation $F(x) = 0$ in the domain $U_0 = U(x_*, \rho_1) \cap U[x_*, \frac{1}{l_1}]$.

Proof. Let $y_* \in U_0$ with $F(x) = 0$. Define the linear operator $S = [x_*, y_*; F]$. By applying the condition (29) and (30), it follows that

$$\|F'(x_*)^{-1}(S-F'(x_*))\| \le l_1\|x_*-y_*\| \le l_1\rho_1 < 1,$$

thus, S^{-1} exists. Then, from the identity $x_* - y_* = S^{-1}(F(x_*) - F(y_*)) = S^{-1}(0)$, we conclude that $y_* = x_*$. □

5. Numerical Examples

In this section, we present numerical examples that confirm obtained semi-local theoretical results.

Firstly, we consider a nonlinear equation. Let $X = \mathbb{R}$, $\Omega = (0.8,\ 1.3)$ and

$$F(x) = x^3 - 1 = 0.$$

Let us determine the Lipschitz constants from conditions (R_2). We can write

$$|F(y) - F(x_0)| = |y^3 - x_0^3| = |y^2 + yx_0 + x_0^2||y - x_0|.$$

It follows that $\lambda = \max_{y\in\Omega}|y^2 + yx_0 + x_0^2|$. For divided difference $[x, y; F]$, we have

$$[x,y;F] = x^2 + xy + y^2$$

and

$$[x,y;F] - [u,v;F] = (x-u)(x+y+u) + (y-v)(y+u+v).$$

We obtain from the last equality that

$$|T_0^{-1}([x,y;F]-[u,y;F])| \le \frac{1}{|T_0|}\max_{x,y,u,v\in\Omega}\{|x+y+u|,|y+u+v|\}(|x-u|+|y-v|).$$

If $a = d$, $b = p$, $c = q$, and F is Fréchet-differentiable, then we obtain methods with derivatives. In this case, $[x, x; F] = F'(x)$ and

$$F'(u) - F'(u_0) = 3(u + u_0)(u - u_0) \;\Rightarrow\; L_0 = \frac{1.5}{|T_0|} \max_{u\in\Omega} |u + u_0|.$$

In Table 1, there are Lipschitz constants from conditions (R_2) and the value s_* to which the sequence $\{t_n\}$ converges. We see that in both cases sequences $\{x_n\}$ is contained in $U(x_0, s_*) \subset \Omega$.

Table 1. Lipschitz constants and radii.

Method	L_0	L	λ	s_*
Newton	0.9917	1.0744	4.3300	0.1023
Secant	1.0101	1.0647	4.3300	0.1129

In Table 2, there are values of the error at each step. The calculations were performed for initial approximation $x_0 = 1.1$ and an accuracy $\varepsilon = 10^{-10}$. For the Secant method, $x_{-1} = 1.11$. We see from the obtained results that

$$|x_n - x_{n-1}| \le t_n - t_{n-1}$$

is performed for each $n \ge 1$.

Table 2. Results for Newton and Secant method.

n	Newton Method			Secant Method		
	x_n	$\lvert x_n - x_{n-1}\rvert$	$t_n - t_{n-1}$	x_n	$\lvert x_n - x_{n-1}\rvert$	$t_n - t_{n-1}$
1	1.0088	9.1185×10^{-2}	9.1185×10^{-2}	1.0096	9.0361×10^{-2}	9.0361×10^{-2}
2	1.0001	8.7386×10^{-3}	1.0905×10^{-2}	1.0009	8.7419×10^{-3}	1.0991×10^{-2}
3	1.0000	7.6802×10^{-5}	1.6022×10^{-4}	1.0000	8.8887×10^{-4}	1.5283×10^{-3}
4	1.0000	5.8989×10^{-9}	3.4595×10^{-8}	1.0000	8.5828×10^{-6}	2.6685×10^{-5}
5	1.0000	0	1.6098×10^{-15}	1.0000	7.7050×10^{-9}	5.7989×10^{-8}
6				1.0000	6.6169×10^{-14}	2.1673×10^{-12}

Then, we consider a system of nonlinear equations. Let $X = \mathbb{R}^3$, $\Omega = U(0,\ 1)$ and

$$F(x) = \begin{pmatrix} e^{x_1} - 1 \\ \frac{e-1}{2} x_2^3 + x_2 \\ x_3 \end{pmatrix} = 0.$$

Since $|e^{t_1} - e^{t_2}| \le e|t_1 - t_2|$, then

$$\lambda = \max\left\{ e, \frac{e-1}{2} \max|y_2^2 + y_2\tau_0 + \tau_0^2| + 1, 1 \right\},\; x_0 = (\xi_0, \tau_0, \rho_0)^T$$

and

$$L_0 = \|T_0^{-1}\| \max\left\{ \frac{e}{2}, \frac{e-1}{2} M_0 \right\}, \quad L = \|T_0^{-1}\| \max\left\{ \frac{e}{2}, \frac{e-1}{2} M \right\}.$$

The constants M_0 and M are calculated similarly to the previous example.

Tables 3 and 4 show results for system of nonlinear equations. The calculations were performed for initial approximations $x_0 = (0.07,\ 0.07,\ 0.07)^T$, $x_{-1} = (0.08,\ 0.08,\ 0.08)^T$ and an accuracy $\varepsilon = 10^{-10}$. From the obtained results we see that

$$\|x_n - x_{n-1}\| \le t_n - t_{n-1}$$

is satisfied for each $n \ge 1$.

Table 3. Lipschitz constants and radii.

Method	L_0	L	λ	s_*
Newton	1.3789	2.5774	2.7183	0.0864
Secant	1.7784	2.5774	2.7183	0.1041

Table 4. Results for Newton and Secant method.

n	Newton Method		Secant Method	
	$\|x_n - x_{n-1}\|$	$t_n - t_{n-1}$	$\|x_n - x_{n-1}\|$	$t_n - t_{n-1}$
1	7.0000×10^{-2}	7.0000×10^{-2}	7.0000×10^{-2}	7.0000×10^{-2}
2	2.3910×10^{-3}	1.5651×10^{-2}	2.6368×10^{-3}	1.7556×10^{-2}
3	2.8629×10^{-6}	8.2657×10^{-4}	9.4309×10^{-5}	5.9446×10^{-3}
4	4.0981×10^{-12}	2.3125×10^{-6}	1.2890×10^{-7}	5.7643×10^{-4}
5			6.0867×10^{-12}	1.5803×10^{-5}

6. Conclusions

A unified convergence analysis of the method without derivatives is provided under the classical Lipschitz conditions for first-order divided differences. The current convergence analysis allows for a comparison between specialized methods that was not possible before under the same set of conditions. The results of the numerical experiment that confirmed the theoretical one are given. The developed technique can also be employed on multipoint as well as multi-step iterative methods [13,14]. This is a possible direction for future areas of research.

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